IMAFF-96/40, NIKHEF-96-008
hep-th/9701041

Families of Singular and Subsingular Vectors of the Topological N=2 Superconformal Algebra

Beatriz Gato-Rivera[a,b] **and Jose Ignacio Rosado**[a]

[a] *Instituto de Matemáticas y Física Fundamental, CSIC, Serrano 123, Madrid 28006, Spain* *

[b]*NIKHEF-H, Kruislaan 409, NL-1098 SJ Amsterdam, The Netherlands*

ABSTRACT

We analyze several issues concerning the singular vectors of the Topological N=2 Superconformal algebra. First we investigate which types of singular vectors exist, regarding the relative U(1) charge and the BRST-invariance properties, finding four different types in chiral Verma modules and twenty-nine different types in complete Verma modules. Then we study the family structure of the singular vectors, every member of a family being mapped to any other member by a chain of simple transformations involving the spectral flows. The families of singular vectors in chiral Verma modules follow a unique pattern (four vectors) and contain subsingular vectors. We write down these families until level 3, identifying the subsingular vectors. The families of singular vectors in complete Verma modules follow infinitely many different patterns, grouped roughly in five main kinds. We present a particularly interesting thirty-eight-member family at levels 3, 4, 5, and 6, as well as the complete set of singular vectors at level 1 (twenty-eight different types). Finally we analyze the Dörrzapf conditions leading to two linearly independent singular vectors of the same type, at the same level in the same Verma module, and we write down four examples of those pairs of singular vectors, which belong to the same thirty-eight-member family.

October 1997

*e-mail address: bgato@pinar1.csic.es

1 Introduction and Notation

In the last few years, singular vectors of infinite dimensional algebras corresponding to conformal field theories are drawing much attention. Far from being empty objects that one simply would like to get rid of, they rather contain an amazing amount of useful information. For example, as a general feature, their decoupling from all other states in the corresponding Verma module gives rise to differential equations which can be solved for correlators of conformal fields. Also, their possible vanishing in the Fock space of the theories is directly connected with the existence of extra states in the Hilbert space that are not primary and not secondary (not included in any Verma modules) [1]. In some specific theories the corresponding singular vectors are, in addition, directly related to Lian-Zuckermann states [2] [3].

Regarding the construction of singular vectors, using either the "fusion" method or the "analytic continuation" method, explicit general expressions have been obtained for the singular vectors of the Virasoro algebra [4], the Sl(2) Kac-Moody algebra [5], the Affine algebra $A_1^{(1)}$ [6], the N=1 Superconformal algebra [7], the Antiperiodic N=2 Superconformal algebra [8], [9], and some W algebras [10]. There is also the method of construction of singular vertex operators, which produce singular vectors when acting on the vacuum [11]. In some cases it is possible to transform singular vectors of an algebra into singular vectors of the same or a different algebra, simplifying notably the computation of the latter ones. For example, Kac-Moody singular vectors have been transformed into Virasoro ones, by using the Knizhnik-Zamolodchikov equation [12], and singular vectors of W algebras have been obtained out of $A_2^{(1)}$ singular vectors via a quantum version of the highest weight Drinfeld-Sokolov gauge transformations [13].

The singular vectors of the Topological N=2 Superconformal algebra have been considered mostly in chiral Verma modules. Some interesting features of such topological singular vectors have appeared in a series of papers, starting in the early nineties. For example, in [14] and [15] it was shown that the uncharged BRST-invariant singular vectors, in the "mirror bosonic string" realization (KM) of the Topological algebra, are related to Virasoro constraints on the KP τ -function. In [16] an isomorphism was uncovered between the uncharged BRST-invariant singular vectors and the singular vectors of the Sl(2) Kac- Moody algebra (this was proved until level four). In [3] the singular vectors were related to Lian-Zuckermann states. Some properties of the singular vectors in the DDK and KM realizations were analyzed in [17]. In [18] the complete set of singular vectors at level 2 (four types in chiral Verma modules) was written down, together with the "universal" spectral flow automorphism $\mathcal{A}$ which transforms all types of topological singular vectors back into singular vectors.

In this paper we investigate several issues related to the singular vectors of the Topo-

logical N=2 Superconformal algebra, denoted as topological singular vectors, considering chiral as well as complete Verma modules. The results are presented as follows. In section 2 we discuss the possible types of topological singular vectors which may exist, taking into account the relative U(1) charge and the BRST-invariance properties of the vector and of the primary state on which it is built. We use an algebraic mechanism, the "cascade effect", which provides a necessary (although not sufficient) condition for the existence of singular vectors of a given type, finding four different types in chiral Verma modules and twenty-nine different types in complete Verma modules. All these types of topological singular vectors exist already at level 1, except one type which only exists at level zero.

In section 3 we analyze a set of mappings which transform topological singular vectors into each other (of a different or of the same Verma module). These mappings give rise to family structures which depend on the types of singular vectors and Verma modules involved.

In section 4 we derive the family structure corresponding to singular vectors in chiral Verma modules. We find a unique structure consisting of four singular vectors, one of each type at the same level, involving generically two different chiral Verma modules. We write down the complete families until level 3. These families contain subsingular vectors; we identify them in the given families and we conjecture an infinite tower of them for higher levels.

In section 5 we derive the family structure corresponding to singular vectors in complete Verma modules. We find an infinite number of different patterns which can be roughly grouped in five main kinds. Then we derive the spectra of conformal weights Δ and U(1) charges h corresponding to the complete Verma modules which contain generic and chiral singular vectors.

In section 6 we analyze some conditions under which the chains of mappings act inside a Verma module, transforming some types of singular vectors into singular vectors of exactly the same types at the same level. We then analyze the Dörrzapf equations, originally written for the Antiperiodic N=2 Superconformal algebra, leading to the existence of two linearly independent singular vectors of the same type, at the same level in the same Verma module. We present examples which prove that some (at least) of those singular vectors are transformed into each other by the mappings described here, *i.e.* they belong to the same families, and we conjecture that the same is true for all of them; that is, that the two partners in each Dörrzapf pair belong to the same family.

Section 7 is devoted to conclusions and final remarks. In Appendix A we describe the "cascade effect", in Appendix B we write down the whole set of singular vectors at level 1 in complete Verma modules, and in Appendix C we present a particularly interesting thirty-eight-member family of singular vectors at levels 3, 4, 5, and 6.

Notation

Highest weight (h.w.) states denote states annihilated by all the positive modes of the generators of the algebra, *i.e.* $\mathcal{L}_{n\geq 1}|\chi\rangle = \mathcal{H}_{n\geq 1}|\chi\rangle = \mathcal{G}_{n\geq 1}|\chi\rangle = \mathcal{Q}_{n\geq 1}|\chi\rangle = 0$.

Primary states denote non-singular h.w. states.

Secondary or descendant states denote states obtained by acting on the h.w. states with the negative modes of the generators of the algebra and with the fermionic zero modes $\mathcal{Q}_0$ and $\mathcal{G}_0$. The fermionic zero modes can also interpolate between two h.w. states at the same footing (two primary states or two singular vectors).

Chiral topological states $|\chi\rangle^{G,Q}$ are states annihilated by both $\mathcal{G}_0$ and $\mathcal{Q}_0$, *i.e.* $\mathcal{G}_0|\chi\rangle^{G,Q} = \mathcal{Q}_0|\chi\rangle^{G,Q} = 0$.

$\mathcal{G}_0$*-closed topological states* $|\chi\rangle^{G}$ denote non-chiral states annihilated by $\mathcal{G}_0$, *i.e.* $\mathcal{G}_0|\chi\rangle^{G} = 0$.

$\mathcal{Q}_0$*-closed topological states* $|\chi\rangle^{Q}$ denote non-chiral states annihilated by $\mathcal{Q}_0$, *i.e.* $\mathcal{Q}_0|\chi\rangle^{Q} = 0$ (they are BRST-invariant since $\mathcal{Q}_0$ is the BRST charge).

$\mathcal{G}_0$*-exact topological states* are $\mathcal{G}_0$-closed or chiral states that can be expressed as the action of $\mathcal{G}_0$ on another state: $|\gamma\rangle = \mathcal{G}_0|\chi\rangle$.

$\mathcal{Q}_0$*-exact topological states* are $\mathcal{Q}_0$-closed or chiral states that can be expressed as the action of $\mathcal{Q}_0$ on another state: $|\gamma\rangle = \mathcal{Q}_0|\chi\rangle$.

No-label topological states $|\chi\rangle$ denote states that cannot be expressed as linear combinations of $\mathcal{G}_0$-closed and $\mathcal{Q}_0$-closed states.

The Verma module associated to a h.w. state consists of the h.w. state plus the set of secondary states built on it. For some Verma modules the h.w. state is degenerate, the fermionic zero modes interpolating between the two h.w. states.

Null vectors are zero-norm states.

Singular vectors are h.w. null vectors.

Generic singular vectors are $\mathcal{G}_0$-closed and $\mathcal{Q}_0$-closed singular vectors built on $\mathcal{G}_0$-closed or $\mathcal{Q}_0$-closed primary states.

Subsingular vectors are non-h.w. null vectors not descendants of any singular vectors, which become singular (*i.e.* h.w.) in the quotient of the Verma module by a submodule generated by singular vectors.

Secondary singular vectors are singular vectors built on singular vectors. The level-zero secondary singular vectors cannot "come back" to the singular vectors on which they are built by acting with $\mathcal{G}_0$ or $\mathcal{Q}_0$.

The Topological N=2 superconformal algebra will be denoted as *the Topological algebra*.

The Antiperiodic N=2 superconformal algebra will be denoted as *the NS algebra*.

The singular vectors of the Topological algebra will be denoted as *topological singular vectors*.

The singular vectors of the NS algebra will be denoted as *NS singular vectors*.

2 Singular Vectors of the Topological Algebra

2.1 Basic Concepts

The Topological algebra

The algebra obtained by applying the topological twists on the NS algebra reads [20]

$$\begin{array}{lcllcll}
[\mathcal{L}_m, \mathcal{L}_n] & = & (m-n)\mathcal{L}_{m+n}\,, & [\mathcal{H}_m, \mathcal{H}_n] & = & \frac{\mathsf{c}}{3} m \delta_{m+n,0}\,, & \\
[\mathcal{L}_m, \mathcal{G}_n] & = & (m-n)\mathcal{G}_{m+n}\,, & [\mathcal{H}_m, \mathcal{G}_n] & = & \mathcal{G}_{m+n}\,, & \\
[\mathcal{L}_m, \mathcal{Q}_n] & = & -n\mathcal{Q}_{m+n}\,, & [\mathcal{H}_m, \mathcal{Q}_n] & = & -\mathcal{Q}_{m+n}\,, & m,\ n \in \mathbf{Z}\,. \\
[\mathcal{L}_m, \mathcal{H}_n] & = & -n\mathcal{H}_{m+n} + \frac{\mathsf{c}}{6}(m^2+m)\delta_{m+n,0}\,, & & & & \\
\{\mathcal{G}_m, \mathcal{Q}_n\} & = & 2\mathcal{L}_{m+n} - 2n\mathcal{H}_{m+n} + \frac{\mathsf{c}}{3}(m^2+m)\delta_{m+n,0}\,, & & & &
\end{array} \quad (2.1)$$

where $\mathcal{L}_m$ and $\mathcal{H}_m$ are the bosonic generators corresponding to the energy momentum tensor (Virasoro generators) and the topological $U(1)$ current respectively, while $\mathcal{Q}_m$ and $\mathcal{G}_m$ are the fermionic generators corresponding to the BRST current and the spin-2 fermionic current respectively. The eigenvalues of $\mathcal{L}_0$ and $\mathcal{H}_0$ correspond to the conformal weight Δ and the U(1) charge h of the states. The “topological” central charge c is the central charge corresponding to the NS algebra. This algebra is topological because the Virasoro generators can be expressed as $\mathcal{L}_m = \frac{1}{2}\{\mathcal{G}_m, \mathcal{Q}_0\}$, where $\mathcal{Q}_0$ is the BRST charge. This implies, as is well known, that the correlators of the fields do not depend on the metric.

Topological twists

The two possible topological twists of the NS superconformal generators are:

$$\begin{array}{lcllcl}
\mathcal{L}^{(1)}_m & = & L_m + \frac{1}{2}(m+1)H_m\,, & & & \\
\mathcal{H}^{(1)}_m & = & H_m\,, & & & \\
\mathcal{G}^{(1)}_m & = & G^+_{m+\frac{1}{2}}\,, & \mathcal{Q}^{(1)}_m & = & G^-_{m-\frac{1}{2}}\,,
\end{array} \quad (2.2)$$

and

$$\begin{aligned}
\mathcal{L}_m^{(2)} &= L_m - \tfrac{1}{2}(m+1)H_m\,, \\
\mathcal{H}_m^{(2)} &= -H_m\,, \\
\mathcal{G}_m^{(2)} &= G^-_{m+\frac{1}{2}}\,, \qquad\qquad \mathcal{Q}_m^{(2)} = G^+_{m-\frac{1}{2}}\,.
\end{aligned} \tag{2.3}$$

These twists, which we denote as T_{W1} and T_{W2}, are mirrored under the interchange $H_m \leftrightarrow -H_m$, $G_r^+ \leftrightarrow G_r^-$. Observe that the h.w. conditions $G^{\pm}_{1/2}\,|\chi_{NS}\rangle = 0$ of the NS algebra read $\mathcal{G}_0|\chi\rangle = 0$ after the corresponding twists. Therefore, any h.w. state of the NS algebra results in a $\mathcal{G}_0$-closed or chiral state of the Topological algebra, which is also h.w. as the reader can easily verify by inspecting the twists (2.2) and (2.3). Conversely, any $\mathcal{G}_0$-closed or chiral h.w. topological state (and only these) transforms into a h.w. state of the NS algebra.

Topological states

From the anticommutator $\{\mathcal{Q}_0, \mathcal{G}_0\} = 2\mathcal{L}_0$ one deduces that a topological state (primary or secondary) with non-zero conformal weight can be either $\mathcal{G}_0$-closed, or $\mathcal{Q}_0$-closed, or a linear combination of both types. One deduces also that $\mathcal{Q}_0$-closed ($\mathcal{G}_0$-closed) topological states with non-zero conformal weight are $\mathcal{Q}_0$-exact ($\mathcal{G}_0$-exact) as well. The topological states with zero conformal weight, however, can be $\mathcal{Q}_0$-closed (satisfying $\mathcal{Q}_0\mathcal{G}_0|\chi\rangle^Q = 0$), or $\mathcal{G}_0$-closed (satisfying $\mathcal{G}_0\mathcal{Q}_0|\chi\rangle^G = 0$), or chiral, or no-label (satisfying $\mathcal{Q}_0\mathcal{G}_0|\chi\rangle = -\mathcal{G}_0\mathcal{Q}_0|\chi\rangle$). Hence, since physical states in string theory are BRST cohomology classes*, only the topological states which are chiral, or $\mathcal{Q}_0$-closed with zero conformal weight, may be physical ($\mathcal{Q}_0$-closed but not $\mathcal{Q}_0$-exact).

Topological primaries

Of special interest are the topological chiral primaries $|0,\mathsf{h}\rangle^{G,Q}$, annihilated by both $\mathcal{G}_0$ and $\mathcal{Q}_0$. Since the conformal weight Δ of these primaries is zero the only quantum number carried by them is the $U(1)$ charge h: $\mathcal{H}_0|0,\mathsf{h}\rangle^{G,Q} = \mathsf{h}|0,\mathsf{h}\rangle^{G,Q}$. Regarding the twists (2.2) and (2.3), a key observation is that $(G^+_{1/2}, G^-_{-1/2})$ results in $(\mathcal{G}_0^{(1)}, \mathcal{Q}_0^{(1)})$ and $(G^-_{1/2}, G^+_{-1/2})$ gives $(\mathcal{G}_0^{(2)}, \mathcal{Q}_0^{(2)})$. Therefore the topological chiral primaries $|\Phi^{(1)}\rangle$ and $|\Phi^{(2)}\rangle$ correspond to the antichiral primaries (*i.e.* $G^-_{-1/2}|\Phi^{(1)}\rangle = 0$) and to the chiral primaries (*i.e.* $G^+_{-1/2}|\Phi^{(2)}\rangle = 0$) of the NS algebra, respectively. In our analysis we will consider also the topological primaries without additional constraints, *i.e.* either $\mathcal{G}_0$-closed or $\mathcal{Q}_0$-closed primaries, denoted as $|\Delta,\,\mathsf{h}\rangle^G$ and $|\Delta,\,\mathsf{h}\rangle^Q$, or no-label primaries, denoted as $|0,\,\mathsf{h}\rangle$. We will not consider primaries $|\Delta,\,\mathsf{h}\rangle$ which can be expressed as linear combinations of $\mathcal{G}_0$-closed and $\mathcal{Q}_0$-closed primaries.

*The reader not familiar with this issue may benefit from reading section 3.2 of ref. [21].

Topological secondaries

As a first classification of the topological secondary states one considers their level l, their *relative* U(1) charge q and their transformation properties under $\mathcal{Q}_0$ and $\mathcal{G}_0$ (BRST-invariance properties). The level l and the relative charge q are defined as the difference between the conformal weight and U(1) charge of the secondary state and the conformal weight and U(1) charge of the primary state on which it is built. Hence the topological secondary states will be denoted as $|\chi\rangle_l^{(q)G}$ ($\mathcal{G}_0$-closed), $|\chi\rangle_l^{(q)Q}$ ($\mathcal{Q}_0$-closed), $|\chi\rangle_l^{(q)G,Q}$ (chiral), and $|\chi\rangle_l^{(q)}$ (no-label). For convenience we will also indicate the conformal weight Δ, the U(1) charge h, and the BRST-invariance properties of the primary state on which the secondary is built. Observe that the conformal weight and the total U(1) charge of the secondary states are given by $\Delta + l$ and $\mathsf{h} + q$, respectively. Thus the chiral and no-label secondary states satisfy $\Delta + l = 0$.

Chiral Verma modules

The Verma modules $V(|0,\mathsf{h}\rangle^{G,Q})$, built on chiral primaries, will be denoted as chiral Verma modules. They are not complete because the chirality constraint is an additional constraint on the primary state not required by the algebra.

Generic Verma modules

The Verma modules $V(|\Delta,\mathsf{h}\rangle^{G})$ and $V(|\Delta,\mathsf{h}\rangle^{Q})$, built on $\mathcal{G}_0$-closed and $\mathcal{Q}_0$-closed primaries without additional constraints, will be denoted as generic Verma modules. They are complete in the sense that the constraints of being annihilated either by $\mathcal{G}_0$ or by $\mathcal{Q}_0$ are required by the algebra. Namely, any state $|\Delta,\mathsf{h}\rangle$ with $\Delta \neq 0$ is either $\mathcal{G}_0$-closed or $\mathcal{Q}_0$-closed or a linear combination of both types. Furthermore for $\Delta \neq 0$ the h.w. state of any generic Verma module is degenerate, *i.e.* there are two primary states. The reason is that the action of $\mathcal{Q}_0$ on $|\Delta,\mathsf{h}\rangle^G$ produces another primary state: $\mathcal{Q}_0|\Delta,\mathsf{h}\rangle^G = |\Delta,\mathsf{h}-1\rangle^Q$, and similarly the action of $\mathcal{G}_0$ on $|\Delta,\mathsf{h}\rangle^Q$ produces another primary state: $\mathcal{G}_0|\Delta,\mathsf{h}\rangle^Q = |\Delta,\mathsf{h}+1\rangle^G$. Therefore, for $\Delta \neq 0$ the Verma modules $V(|\Delta,\mathsf{h}\rangle^G)$ and $V(|\Delta,\mathsf{h}\rangle^Q)$, built on the primaries $|\Delta,\mathsf{h}\rangle^G$ and $|\Delta,\mathsf{h}\rangle^Q$, are equivalent to the Verma modules $V(|\Delta,\mathsf{h}-1\rangle^Q)$ and $V(|\Delta,\mathsf{h}+1\rangle^G)$, built on the primaries $|\Delta,\mathsf{h}-1\rangle^Q$ and $|\Delta,\mathsf{h}+1\rangle^G$, respectively. For $\Delta = 0$, however, $\mathcal{Q}_0|0,\mathsf{h}\rangle^G$ and $\mathcal{G}_0|0,\mathsf{h}\rangle^Q$ are not primary states but level-zero chiral singular vectors instead, denoted as $|\chi\rangle_{0,|0,\mathsf{h}\rangle^G}^{(-1)G,Q}$ and $|\chi\rangle_{0,|0,\mathsf{h}\rangle^Q}^{(1)G,Q}$ respectively, so that the h.w. states of the Verma modules $V(|0,\mathsf{h}\rangle^G)$ and $V(|0,\mathsf{h}\rangle^Q)$ are not degenerate.

No-label Verma modules

The Verma modules $V(|0,\mathsf{h}\rangle)$, built on no-label primaries, will be denoted as no-label Verma modules. They are complete, obviously, since the no-label primaries are annihilated only by the positive modes of the generators of the algebra.

2.2 Types of topological singular vectors

Let us discuss which types of topological secondary states can be singular, a given type being defined by the relative charge q, the BRST-invariance properties of the state itself, and the BRST-invariance properties of the primary on which it is built.

First of all, we will consider only four different types of topological primaries: $|\Delta, \mathsf{h}\rangle^G$, $|\Delta, \mathsf{h}\rangle^Q$ and $|0, \mathsf{h}\rangle$, without additional constraints, and $|0, \mathsf{h}\rangle^{G,Q}$ with the chirality constraint. Therefore we will not consider primaries which are linear combinations of two or more of these types, neither primaries of these types with additional constraints (other than the chirality constraint) giving rise to incomplete Verma modules.

As to the secondary states $|\chi\rangle_l^{(q)G}$, $|\chi\rangle_l^{(q)Q}$, $|\chi\rangle_l^{(q)G,Q}$ and $|\chi\rangle_l^{(q)}$ (the latter two with zero conformal weight, *i.e.* satisfying $\Delta + l = 0$), taking into account that there are four kinds of topological primaries to be considered, a naive estimate would give sixteen different types of them for every allowed value of the relative charge q, which in turn is determined by the level l. This is incorrect, however, because the chiral primaries $|0, \mathsf{h}\rangle^{G,Q}$ have no secondaries of types $|\chi\rangle_l^{(q)G,Q}$ and $|\chi\rangle_l^{(q)}$, and the no-label primaries $|0, \mathsf{h}\rangle$ have no secondaries of type $|\chi\rangle_l^{(q)}$, and have a secondary of type $|\chi\rangle_l^{(q)G,Q}$ only for $l = 0,\ q = 0$. Observe that there are no chiral secondary states in chiral Verma modules.

Thus there are twelve different types of secondary states, with well defined BRST-invariance properties, for every non-zero allowed value of q, and thirteen types for $q = 0$ (although the extra type only exists for level zero). However, only a few of these types admit singular vectors. The reason is that, when one imposes the h.w. conditions, the allowed values of q reduce drastically and, in addition, not all the twelve types do exist for a given non-zero allowed value of q, as we will see.

The question now arises as whether any singular vector with non-zero conformal weight can be decomposed into a $\mathcal{G}_0$-closed singular vector plus a $\mathcal{Q}_0$-closed singular vector. From the anticommutator $\{\mathcal{Q}_0, \mathcal{G}_0\} = 2\mathcal{L}_0$ one obtains the decomposition

$$|\chi\rangle_l = \frac{1}{2(\Delta + l)}\ \mathcal{G}_0\mathcal{Q}_0|\chi\rangle_l + \frac{1}{2(\Delta + l)}\ \mathcal{Q}_0\mathcal{G}_0|\chi\rangle_l = |\chi\rangle_l^G + |\chi\rangle_l^Q\,. \tag{2.4}$$

If $|\chi\rangle_l$ is a singular vector, *i.e.* satisfies the h.w. conditions $\mathcal{L}_{n\geq1}|\chi\rangle = \mathcal{H}_{n\geq1}|\chi\rangle = \mathcal{G}_{n\geq1}|\chi\rangle = \mathcal{Q}_{n\geq1}|\chi\rangle = 0$, then $\mathcal{G}_0\mathcal{Q}_0|\chi\rangle_l$ and $\mathcal{Q}_0\mathcal{G}_0|\chi\rangle_l$ satisfy the h.w. conditions too, as one deduces straightforwardly using the Topological algebra (2.1). Therefore, regarding singular vectors with non-zero conformal weight, we can restrict ourselves to $\mathcal{G}_0$-closed and to $\mathcal{Q}_0$-closed singular vectors.

We have not found a rigorous method so far to deduce which types of singular vectors do exist. We have identified an algebraic mechanism, however, which is a key fact

underlying whether or not a given type of topological secondary state admits singular vectors. This mechanism, which we denote "the cascade effect", consists of the vanishing in cascade of the coefficients of the "would-be" singular vector when the h.w. conditions are imposed, alone or in combination with the BRST and/or anti-BRST-invariance conditions $\mathcal{Q}_0|\chi\rangle = 0$ and/or $\mathcal{G}_0|\chi\rangle = 0$. As is explained in Appendix A, the starting of the cascade effect, which occurs in most of the possible types, is very easy to determine. Once it starts, the cascade effect goes on until the end, getting rid of all the coefficients of the would-be singular vectors. As a result the types of secondary states for which the cascade effect takes place do not admit singular vectors. The rigorous proofs of these statements will be presented in a forthcoming paper [22].

The cascade effect takes place in all cases for $|q| > 2$, and also for $|q| = 2$ in chiral Verma modules. The types of singular vectors allowed by the cascade effect are:

- Four types built on chiral primaries $|0, \mathsf{h}\rangle^{G,Q}$:

$$\begin{array}{r|lll} & q=-1 & q=0 & q=1 \\ \hline \mathcal{G}_0\text{-closed} & - & |\chi\rangle_l^{(0)G} & |\chi\rangle_l^{(1)G} \\ \mathcal{Q}_0\text{-closed} & |\chi\rangle_l^{(-1)Q} & |\chi\rangle_l^{(0)Q} & - \end{array} \tag{2.5}$$

- Ten types built on $\mathcal{G}_0$-closed primaries $|\Delta, \mathsf{h}\rangle^{G}$:

$$\begin{array}{r|llll} & q=-2 & q=-1 & q=0 & q=1 \\ \hline \mathcal{G}_0\text{-closed} & - & |\chi\rangle_l^{(-1)G} & |\chi\rangle_l^{(0)G} & |\chi\rangle_l^{(1)G} \\ \mathcal{Q}_0\text{-closed} & |\chi\rangle_l^{(-2)Q} & |\chi\rangle_l^{(-1)Q} & |\chi\rangle_l^{(0)Q} & - \\ \text{chiral} & - & |\chi\rangle_l^{(-1)G,Q} & |\chi\rangle_l^{(0)G,Q} & - \\ \text{no-label} & - & |\chi\rangle_l^{(-1)} & |\chi\rangle_l^{(0)} & - \end{array} \tag{2.6}$$

- Ten types built on $\mathcal{Q}_0$-closed primaries $|\Delta, \mathsf{h}\rangle^{Q}$:

$$\begin{array}{r|llll} & q=-1 & q=0 & q=1 & q=2 \\ \hline \mathcal{G}_0\text{-closed} & - & |\chi\rangle_l^{(0)G} & |\chi\rangle_l^{(1)G} & |\chi\rangle_l^{(2)G} \\ \mathcal{Q}_0\text{-closed} & |\chi\rangle_l^{(-1)Q} & |\chi\rangle_l^{(0)Q} & |\chi\rangle_l^{(1)Q} & - \\ \text{chiral} & - & |\chi\rangle_l^{(0)G,Q} & |\chi\rangle_l^{(1)G,Q} & - \\ \text{no-label} & - & |\chi\rangle_l^{(0)} & |\chi\rangle_l^{(1)} & - \end{array} \tag{2.7}$$

- Nine types built on no-label primaries $|0, \mathsf{h}\rangle$:

$$
\begin{array}{r|lllll}
 & q=-2 & q=-1 & q=0 & q=1 & q=2 \\
\hline
\mathcal{G}_0\text{-closed} & - & |\chi\rangle_l^{(-1)G} & |\chi\rangle_l^{(0)G} & |\chi\rangle_l^{(1)G} & |\chi\rangle_l^{(2)G} \\
\mathcal{Q}_0\text{-closed} & |\chi\rangle_l^{(-2)Q} & |\chi\rangle_l^{(-1)Q} & |\chi\rangle_l^{(0)Q} & |\chi\rangle_l^{(1)Q} & - \\
\text{chiral} & - & - & |\chi\rangle_0^{(0)G,Q} & - & -
\end{array}
\tag{2.8}
$$

The chiral and no-label singular vectors satisfy $\Delta + l = 0$. Observe that the chiral type on no-label primaries only exists for level zero. It is given by $|\chi\rangle_{0,|0,\mathsf{h}\rangle}^{(0)G,Q} = \mathcal{G}_0\mathcal{Q}_0|0,\mathsf{h}\rangle$.

The cascade effect allows therefore four different types of singular vectors in chiral Verma modules and twenty-nine different types in complete Verma modules. It turns out that all these types of singular vectors exist at low levels, as we will see, in spite of the fact that the cascade effect provides a necessary, but not sufficent, condition for their existence. In fact all these types can be constructed at level 1, except the type that only exists at level zero. In addition, the generic and the chiral singular vectors must necessarily exist, as we will explain.

An important observation is that some types of topological singular vectors admit two-dimensional spaces (see section 6). That is, in some Verma modules one finds two linearly independent singular vectors of the same type at the same level. In those cases our notation does not distinguish betwen the two singular vectors and an additional label is necessary. The rigorous analysis of this issue is beyond the scope of this paper and will be performed in a next publication [22].

Generic singular vectors

There are twelve types of generic singular vectors, *i.e.* $\mathcal{G}_0$-closed and $\mathcal{Q}_0$-closed singular vectors in generic Verma modules, as can be seen in tables (2.6) and (2.7). These singular vectors are directly related to the singular vectors of the NS algebra, via the topological twists and the mappings that we will analyze in next section and in section 5. Namely, four of the generic singular vectors ($|\chi\rangle_{l,|\Delta,\mathsf{h}\rangle^G}^{(0)G}$, $|\chi\rangle_{l,|\Delta,\mathsf{h}\rangle^Q}^{(1)G}$, $|\chi\rangle_{l,|\Delta,\mathsf{h}\rangle^G}^{(-1)Q}$, $|\chi\rangle_{l,|\Delta,\mathsf{h}\rangle^Q}^{(0)Q}$) can be mapped to the uncharged NS singular vectors, whereas the remaining eight types ($|\chi\rangle_{l,|\Delta,\mathsf{h}\rangle^G}^{(-2)G}$, $|\chi\rangle_{l,|\Delta,\mathsf{h}\rangle^Q}^{(-1)G}$, $|\chi\rangle_{l,|\Delta,\mathsf{h}\rangle^G}^{(0)Q}$, $|\chi\rangle_{l,|\Delta,\mathsf{h}\rangle^Q}^{(1)Q}$, $|\chi\rangle_{l,|\Delta,\mathsf{h}\rangle^G}^{(-1)G}$, $|\chi\rangle_{l,|\Delta,\mathsf{h}\rangle^Q}^{(0)G}$, $|\chi\rangle_{l,|\Delta,\mathsf{h}\rangle^G}^{(1)G}$, $|\chi\rangle_{l,|\Delta,\mathsf{h}\rangle^Q}^{(2)G}$) can be mapped to the charged NS singular vectors. This argument shows that the generic types of topological singular vectors must necessarily exist.

Chiral singular vectors

An important observation is that chiral singular vectors $|\chi\rangle_l^{(q)G,Q}$ can be regarded as particular cases of $\mathcal{G}_0$-closed singular vectors $|\chi\rangle_l^{(q)G}$ and also as particular cases of $\mathcal{Q}_0$-closed singular vectors $|\chi\rangle_l^{(q)Q}$. That is, some $\mathcal{G}_0$-closed and $\mathcal{Q}_0$-closed singular vectors

may "become" chiral (although not necessarily) when the conformal weight of the singular vector turns out to be zero, *i.e.* $\Delta+l=0$. Notice, however, that there are singular vectors of several types, for example $|\chi\rangle^{(1)G}_{l,|\Delta,\mathsf{h}\rangle^G}$, that cannot "become" chiral since the cascade effect prevents the existence of chiral singular vectors of the corresponding types, for example $|\chi\rangle^{(1)G,Q}_{l,|\Delta,\mathsf{h}\rangle^G}$. Observe also that there are no chiral singular vectors in chiral Verma modules. The chiral singular vectors are related to the singular vectors of the NS algebra, via the untwistings and the mappings, like the generic singular vectors.

Equivalent types of singular vectors

It happens that not all the types of singular vectors shown in tables (2.6) and (2.7) are inequivalent since, for $\Delta \neq 0$, the primaries of types $|\Delta,\mathsf{h}\rangle^G$ and $|\Delta,\mathsf{h}\rangle^Q$ can be mapped into each other inside the same Verma module, as we explained, producing a modification of ± 1 in the U(1) charges h and q of the singular vectors, so that the total U(1) charge remains the same. For example, the singular vectors $|\chi\rangle^{(q)G}_{l,|\Delta,\mathsf{h}\rangle^G}$ are equivalent (for $\Delta \neq 0$) to the singular vectors $|\chi\rangle^{(q+1)G}_{l,|\Delta,\mathsf{h}-1\rangle^Q}$ if we express $|\Delta,\mathsf{h}\rangle^G = \mathcal{G}_0|\Delta,\mathsf{h}-1\rangle^Q$. Furthermore it turns out that some of these types of singular vectors only exist for $\Delta \neq 0$. Namely all four types of no-label singular vectors in generic Verma modules, and the uncharged chiral singular vectors $|\chi\rangle^{(0)G,Q}_{l,|\Delta,\mathsf{h}\rangle^G}$ and $|\chi\rangle^{(0)G,Q}_{l,|\Delta,\mathsf{h}\rangle^Q}$. As a result, the four types of no-label singular vectors reduce to two inequivalent types, and the uncharged chiral singular vectors can be expressed as charged chiral singular vectors $|\chi\rangle^{(1)G,Q}_{l,|\Delta,\mathsf{h}-1\rangle^Q}$ and $|\chi\rangle^{(-1)G,Q}_{l,|\Delta,\mathsf{h}+1\rangle^G}$ respectively. The latter do exist for $\Delta=0$, however.

An important observation is that certain spectral flow mappings distinguish between some types of singular vectors and the equivalent types. Therefore, for practical purposes the pairs of equivalent singular vectors must be taken into account separately (this issue will be discussed in next section and in section 5).

3 Mappings between Topological Singular Vectors

Inside a given Verma module $V(\Delta,\mathsf{h})$ and for a given level l the topological singular vectors are connected by the action of $\mathcal{Q}_0$ and $\mathcal{G}_0$ in the following way:

$$\mathcal{Q}_0|\chi\rangle^{(q)G}_l \to |\chi\rangle^{(q-1)Q}_l \quad , \qquad \mathcal{G}_0|\chi\rangle^{(q)Q}_l \to |\chi\rangle^{(q+1)G}_l \tag{3.1}$$

$$\mathcal{Q}_0|\chi\rangle^{(q)}_l \to |\chi\rangle^{(q-1)Q}_l \quad , \qquad \mathcal{G}_0|\chi\rangle^{(q)}_l \to |\chi\rangle^{(q+1)G}_l \tag{3.2}$$

These arrows can be reversed (up to constants), using $\mathcal{G}_0$ and $\mathcal{Q}_0$ respectively, only if the conformal weight of the singular vectors is different from zero, *i.e.* $\Delta+l \neq 0$. Otherwise,

on the right-hand side of the arrows one obtains secondary singular vectors which cannot "come back" to the singular vectors on the left-hand side and are at level zero with respect to these. In particular this always happens to no-label singular vectors $|\chi\rangle_l^{(q)}$, in the second row, since they always satisfy $\Delta + l = 0$, while this never happens to singular vectors in chiral Verma modules, for which $0 + l > 0$.

Hence $\mathcal{G}_0$ and $\mathcal{Q}_0$ interpolate between two singular vectors with non-zero conformal weight, in both directions, whereas they produce secondary singular vectors when acting on singular vectors with zero conformal weight.

In what follows we will discuss the action of the spectral flow transformations which map topological singular vectors back to topological singular vectors, interpolating between different Verma modules. A very detailed analysis of the spectral flows has been made recently in ref. [19], from which we borrow the notation and some of the results.

The universal odd spectral flow $\mathcal{A}$

The universal odd spectral flow automorphism $\mathcal{A}_1$, denoted simply as $\mathcal{A}$, transforms all kinds of primary states and singular vectors back into primary states and singular vectors, mapping chiral states to chiral states. It is given by [18] [19]

$$\begin{aligned} \mathcal{A}\,\mathcal{L}_m\,\mathcal{A}^{-1} &= \mathcal{L}_m - m\mathcal{H}_m\,, \\ \mathcal{A}\,\mathcal{H}_m\,\mathcal{A}^{-1} &= -\mathcal{H}_m - \tfrac{\mathsf{c}}{3}\delta_{m,0}\,, \\ \mathcal{A}\,\mathcal{Q}_m\,\mathcal{A}^{-1} &= \mathcal{G}_m\,, \\ \mathcal{A}\,\mathcal{G}_m\,\mathcal{A}^{-1} &= \mathcal{Q}_m\,. \end{aligned} \tag{3.3}$$

with $\mathcal{A}^{-1} = \mathcal{A}$. It transforms the $(\mathcal{L}_0, \mathcal{H}_0)$ eigenvalues (Δ, h) of the states as $(\Delta, -\mathsf{h} - \frac{\mathsf{c}}{3})$, reversing the relative charge of the secondary states and leaving the level invariant, as a consequence. In addition, $\mathcal{A}$ also reverses the BRST-invariance properties of the states (primary as well as secondary) mapping $\mathcal{G}_0$-closed ($\mathcal{Q}_0$-closed) states into $\mathcal{Q}_0$-closed ($\mathcal{G}_0$-closed) states, and chiral states into chiral states.

For chiral Verma modules the action of $\mathcal{A}$ results therefore in the mappings [18]

$$\mathcal{A}\,|\chi\rangle_{l,\mathsf{h}}^{(q)Q} \rightarrow |\chi\rangle_{l,-\mathsf{h}-\frac{\mathsf{c}}{3}}^{(-q)G}\,, \qquad \mathcal{A}\,|\chi\rangle_{l,\mathsf{h}}^{(q)G} \rightarrow |\chi\rangle_{l,-\mathsf{h}-\frac{\mathsf{c}}{3}}^{(-q)Q}\,, \tag{3.4}$$

where h and $-\mathsf{h} - \frac{\mathsf{c}}{3}$ denote the chiral primaries $|0, \mathsf{h}\rangle^{G,Q}$ and $|0, -\mathsf{h} - \frac{\mathsf{c}}{3}\rangle^{G,Q}$, respectively. Observe that these two mappings are each other's inverse.

For complete Verma modules the action of $\mathcal{A}$ results in the mappings:

$$\mathcal{A}\,|\chi\rangle_{l,|\Delta,\mathsf{h}\rangle^G}^{(q)G} \rightarrow |\chi\rangle_{l,|\Delta,-\mathsf{h}-\frac{\mathsf{c}}{3}\rangle^Q}^{(-q)Q}\,, \qquad \mathcal{A}\,|\chi\rangle_{l,|\Delta,\mathsf{h}\rangle^G}^{(q)Q} \rightarrow |\chi\rangle_{l,|\Delta,-\mathsf{h}-\frac{\mathsf{c}}{3}\rangle^Q}^{(-q)G}\,,$$

$$\mathcal{A}\,|\chi\rangle^{(q)G,Q}_{l,\,|{-l},\,\mathsf{h}\rangle^G} \rightarrow |\chi\rangle^{(-q)G,Q}_{l,\,|{-l},-\mathsf{h}-\frac{\mathsf{c}}{3}\rangle^Q}\ , \qquad \mathcal{A}\,|\chi\rangle^{(q)}_{l,\,|{-l},\,\mathsf{h}\rangle^G} \rightarrow |\chi\rangle^{(-q)}_{l,\,|{-l},-\mathsf{h}-\frac{\mathsf{c}}{3}\rangle^Q},$$
$$\mathcal{A}\,|\chi\rangle^{(q)G}_{l,\,|0,\,\mathsf{h}\rangle} \rightarrow |\chi\rangle^{(-q)Q}_{l,\,|0,-\mathsf{h}-\frac{\mathsf{c}}{3}\rangle}\ , \qquad \mathcal{A}\,|\chi\rangle^{(q)Q}_{l,\,|0,\,\mathsf{h}\rangle} \rightarrow |\chi\rangle^{(-q)G}_{l,\,|0,-\mathsf{h}-\frac{\mathsf{c}}{3}\rangle}, \tag{3.5}$$

and their inverses.

The universal even-odd spectral flow $\hat{\mathcal{U}}_{\pm1}$

As was explained in refs. [18], [19] the action of the even-odd spectral flow operators $\hat{\mathcal{U}}_{\pm1}$ on the Topological algebra is identical to the action of $\mathcal{A}$, except for the fact that $\hat{\mathcal{U}}_{\pm1}$ connect the two sets of topological generators corresponding to the two different twists of the NS generators, given by (2.2) and (2.3), *i.e.*

$$\begin{aligned}
\hat{\mathcal{U}}_1\,\mathcal{L}^{(2)}_m\,\hat{\mathcal{U}}_1^{-1} &= \mathcal{L}^{(1)}_m - m\mathcal{H}^{(1)}_m\,,\\
\hat{\mathcal{U}}_1\,\mathcal{H}^{(2)}_m\,\hat{\mathcal{U}}_1^{-1} &= -\mathcal{H}^{(1)}_m - \tfrac{\mathsf{c}}{3}\delta_{m,0}\,,\\
\hat{\mathcal{U}}_1\,\mathcal{Q}^{(2)}_m\,\hat{\mathcal{U}}_1^{-1} &= \mathcal{G}^{(1)}_m\,,\\
\hat{\mathcal{U}}_1\,\mathcal{G}^{(2)}_m\,\hat{\mathcal{U}}_1^{-1} &= \mathcal{Q}^{(1)}_m\,.
\end{aligned}\tag{3.6}$$

with $\hat{\mathcal{U}}_1^{-1} = \hat{\mathcal{U}}_{-1}$. One can see this also using the composition rules of the spectral flows [19]. One finds $\hat{\mathcal{U}}_1 = \mathcal{A}\hat{\mathcal{A}}_0$, $\hat{\mathcal{U}}_{-1} = \hat{\mathcal{A}}_0\mathcal{A}$, where $\hat{\mathcal{A}}_0$ is the operator that interchanges the labels $(1) \leftrightarrow (2)$ corresponding to the two sets of topological generators.

The even spectral flow $\mathcal{U}_{\pm1}$

The topological even spectral flow $\mathcal{U}_\theta$ is not universal for any value of θ. In particular it does not map chiral primary states back to chiral primary states. However, for $\theta = 1$ it maps h.w. states annihilated by $\mathcal{G}_0$ into h.w. states annihilated by $\mathcal{Q}_0$, and the other way around for $\theta = -1$. As a result $\mathcal{U}_1$ transforms the singular vectors of types $|\chi\rangle^{(q)G}_{l,|\phi\rangle^G}$ and $|\chi\rangle^{(q)G,Q}_{l,|\phi\rangle^G}$ into singular vectors of types $|\chi\rangle^{(q)Q}_{l,|\phi\rangle^Q}$ and $|\chi\rangle^{(q)G,Q}_{l,|\phi\rangle^Q}$ (both), while mapping all other types of singular vectors to various kinds of states which are not singular vectors. The topological even spectral flow† is given by [23] [19]

$$\begin{aligned}
\mathcal{U}_\theta\,\mathcal{L}_m\,\mathcal{U}_\theta^{-1} &= \mathcal{L}_m + \theta\mathcal{H}_m + \tfrac{\mathsf{c}}{6}(\theta+\theta^2)\delta_{m,0}\,,\\
\mathcal{U}_\theta\,\mathcal{H}_m\,\mathcal{U}_\theta^{-1} &= \mathcal{H}_m + \tfrac{\mathsf{c}}{3}\theta\delta_{m,0}\,,\\
\mathcal{U}_\theta\,\mathcal{G}_m\,\mathcal{U}_\theta^{-1} &= \mathcal{G}_{m+\theta}\,,\\
\mathcal{U}_\theta\,\mathcal{Q}_m\,\mathcal{U}_\theta^{-1} &= \mathcal{Q}_{m-\theta}\,,
\end{aligned}\tag{3.7}$$

and satisfies $\mathcal{U}_\theta^{-1} = \mathcal{U}_{(-\theta)}$. It transforms the $(\mathcal{L}_0, \mathcal{H}_0)$ eigenvalues (Δ, h) of the primary states as $(\Delta - \theta\mathsf{h} + \frac{\mathsf{c}}{6}(\theta^2-\theta), \mathsf{h} - \frac{\mathsf{c}}{3}\theta)$, and the level of the secondary states as $l \rightarrow l - \theta q$, letting invariant the relative charge q. Under $\mathcal{U}_1$ a $\mathcal{G}_0$-closed, or chiral, singular vector at

†This spectral flow was written for the first time in [23], although only in [19] it has been analyzed properly.

level l with relative charge q, built on a primary of type $|\Delta, \mathsf{h}\rangle^G$, is transformed into a $\mathcal{Q}_0$-closed singular vector at level $l-q$ with relative charge q, built on a primary of type $|\Delta - \mathsf{h}, \mathsf{h} - \frac{\mathsf{c}}{3}\rangle^Q$. This $\mathcal{Q}_0$-closed singular vector may "become" chiral if $\Delta - \mathsf{h} = q - l$, and is certainly chiral if the $\mathcal{G}_0$-closed singular vector was annihilated by $\mathcal{G}_{-1}$. Therefore there are four possibilities:

$$|\chi\rangle^{(q)G}_{l,\,|\Delta,\,\mathsf{h}\rangle^G} \xrightarrow{\mathcal{U}_1} |\chi\rangle^{(q)Q}_{l-q,\,|\Delta-\mathsf{h},\,\mathsf{h}-\frac{\mathsf{c}}{3}\rangle^Q}\ , \qquad |\chi\rangle^{(q)G,Q}_{l,\,|-l,\,\mathsf{h}\rangle^G} \xrightarrow{\mathcal{U}_1} |\chi\rangle^{(q)Q}_{l-q,\,|-l-\mathsf{h},\,\mathsf{h}-\frac{\mathsf{c}}{3}\rangle^Q}\ , \tag{3.8}$$

$$|\chi\rangle^{(q)G}_{l,\,|\Delta,\,\mathsf{h}\rangle^G} \xrightarrow{\mathcal{U}_1} |\chi\rangle^{(q)G,Q}_{l-q,\,|q-l,\,\mathsf{h}-\frac{\mathsf{c}}{3}\rangle^Q}\ , \qquad |\chi\rangle^{(0)G,Q}_{l,\,|-l,\,0\rangle^G} \xrightarrow{\mathcal{U}_1} |\chi\rangle^{(0)G,Q}_{l,\,|-l,\,-\frac{\mathsf{c}}{3}\rangle^Q}\ . \tag{3.9}$$

We see that chiral singular vectors are transformed, in most cases, into non-chiral singular vectors (although annihilated by $\mathcal{Q}_{-1}$, as the reader can easily verify). The mapping to another chiral singular vector only occurs under very restricted conditions:[‡] $q = 0$, $\mathsf{h} = 0$, and the requirement that the singular vector is annihilated by $\mathcal{G}_{-1}$. The resulting chiral singular vector, in turn, is annihilated by $\mathcal{Q}_{-1}$.

These are the only types of singular vectors transformed under $\mathcal{U}_1$ into singular vectors, and the other way around using the inverse $\mathcal{U}_{-1}$. The chiral primaries $|0, \mathsf{h}\rangle^{G,Q}$ are transformed under $\mathcal{U}_1$ (or $\mathcal{U}_{-1}$) into non-chiral primaries annihilated by $\mathcal{Q}_{-1}$ (or $\mathcal{G}_{-1}$). Primaries with such constraints, which generate incomplete Verma modules, are beyond the scope of this paper.

Observe that $\mathcal{U}_{\pm 1}$ distinguish between the different types of singular vectors drastically. Regarding the pairs of equivalent singular vectors for $\Delta \neq 0$, $|\chi\rangle^{(q)G}_{l,\,|\Delta,\,\mathsf{h}\rangle^G}$ and $|\chi\rangle^{(q+1)G}_{l,\,|\Delta,\,\mathsf{h}-1\rangle^Q}$ on the one side, and $|\chi\rangle^{(q)Q}_{l,\,|\Delta,\,\mathsf{h}\rangle^Q}$ and $|\chi\rangle^{(q-1)Q}_{l,\,|\Delta,\,\mathsf{h}+1\rangle^G}$ on the other side, $\mathcal{U}_1$ and $\mathcal{U}_{-1}$, respectively, transform only one member of the pair into a singular vector. The situation is more involved for the pairs of equivalent singular vectors which are chiral. Namely, $\mathcal{U}_1$ transforms the chiral singular vector $|\chi\rangle^{(0)G,Q}_{l,\,|-l,\,\mathsf{h}\rangle^G}$ into a singular vector of type $|\chi\rangle^{(0)Q}_{l,\,|-l-\mathsf{h},\,\mathsf{h}-\frac{\mathsf{c}}{3}\rangle^Q}$ while $\mathcal{U}_{-1}$ transforms the equivalent singular vector $|\chi\rangle^{(1)G,Q}_{l,\,|-l,\,\mathsf{h}-1\rangle^Q}$ into a singular vector of type $|\chi\rangle^{(1)G}_{l+1,\,|-l+\mathsf{h}-1+\frac{\mathsf{c}}{3},\,\mathsf{h}-1+\frac{\mathsf{c}}{3}\rangle^G}$, and similarly with the pair of equivalent chiral singular vectors $|\chi\rangle^{(0)G,Q}_{l,\,|-l,\,\mathsf{h}\rangle^Q}$ and $|\chi\rangle^{(-1)G,Q}_{l,\,|-l,\,\mathsf{h}+1\rangle^G}$.

Other spectral flow transformations

The topological odd spectral flow $\mathcal{A}_\theta$ [19] transforms certain types of singular vectors back into singular vectors for $\theta = 0$ and $\theta = 2$, in addition to $\theta = 1$ which corresponds to the universal mapping: $\mathcal{A}_1 = \mathcal{A}$. However, the action of $\mathcal{A}_0$ and $\mathcal{A}_2$ is already included in the composition of $\mathcal{A}$ with $\mathcal{U}_{\pm1}$. Namely, $\mathcal{A}_0 = \mathcal{A}\,\mathcal{U}_1$ and $\mathcal{A}_2 = \mathcal{U}_1\mathcal{A}$.

[‡]There are no chiral singular vectors of types $|\chi\rangle^{(1)G,Q}_{l,\,|\Delta,\,\mathsf{h}\rangle^G}$ and $|\chi\rangle^{(-1)G,Q}_{l,\,|\Delta,\,\mathsf{h}\rangle^Q}$.

Similarly, the even-odd spectral flow transformations $\hat{\mathcal{U}}_0$ and $\hat{\mathcal{U}}_{\pm 2}$ [19] map some types of topological singular vectors to singular vectors, but their action is identical to the action of $\mathcal{A}_0$ and $\mathcal{A}_2$, although interchaging the labels (1) $\leftrightarrow$ (2) of the two sets of topological generators: $\hat{\mathcal{U}}_0 = \mathcal{A}_0\hat{\mathcal{A}}_0$, $\hat{\mathcal{U}}_2 = \mathcal{A}_2\hat{\mathcal{A}}_0$ and $\hat{\mathcal{U}}_{-2} = \hat{\mathcal{A}}_0\mathcal{A}_2$, where $\hat{\mathcal{A}}_0$ is the label-exchange operator.

Finally, the odd-even spectral flow transformations $\hat{\mathcal{A}}_{\pm 1}$ [19] act like $\mathcal{U}_{\pm 1}$ but interchanging the labels (1) $\leftrightarrow$ (2) since $\hat{\mathcal{A}}_{\pm 1} = \mathcal{U}_{\pm 1}\hat{\mathcal{A}}_0$.

4 Families of Topological Singular Vectors in Chiral Verma Modules

4.1 Family Structure

Now let us apply the results of sections 2 and 3 to analyze the family structure of the topological singular vectors in chiral Verma modules, *i.e.* built on chiral primaries $|0,\mathsf{h}\rangle^{G,Q}$. We found that in chiral Verma modules the cascade effect allows only four types of topological singular vectors: $|\chi\rangle^{(0)G}$, $|\chi\rangle^{(0)Q}$, $|\chi\rangle^{(1)G}$ and $|\chi\rangle^{(-1)Q}$. These four types of singular vectors have been constructed at low levels (see subsection 4.4). They are connected to each other by the action of $\mathcal{Q}_0$, $\mathcal{G}_0$ and $\mathcal{A}$, at the same level l, in the way shown by the diagram:

$$\begin{array}{ccc} |\chi\rangle^{(0)G}_{l,\mathsf{h}} & \xrightarrow{\mathcal{Q}_0} & |\chi\rangle^{(-1)Q}_{l,\mathsf{h}} \\ \mathcal{A}\updownarrow & & \updownarrow\mathcal{A} \\ |\chi\rangle^{(0)Q}_{l,-\mathsf{h}-\frac{\mathsf{c}}{3}} & \xrightarrow{\mathcal{G}_0} & |\chi\rangle^{(1)G}_{l,-\mathsf{h}-\frac{\mathsf{c}}{3}} \end{array} \tag{4.1}$$

Hence the topological singular vectors built on chiral primaries $|0,\mathsf{h}\rangle^{G,Q}$ come in families of four: one of each kind at the same level. Two of them, one charged and one uncharged, belong to the chiral Verma module $V(|0,\mathsf{h}\rangle^{G,Q})$, whereas the other pair belong to a different chiral Verma module $V(|0,-\mathsf{h}-\frac{\mathsf{c}}{3}\rangle^{G,Q})$. This implies that it is sufficient to compute only one of these four singular vectors from scratch, the other three being generated by the action of $\mathcal{G}_0$, $\mathcal{Q}_0$ and $\mathcal{A}$. For $\mathsf{h} = -\frac{\mathsf{c}}{6}$ the two chiral Verma modules related by $\mathcal{A}$ coincide. Therefore, if there are singular vectors for this value of h (see next

subsection), they must come four by four *in the same* chiral Verma module: one of each kind at the same level.

As was explained in section 3, the action of $\hat{\mathcal{U}}_{\pm 1}$ (3.6), which is the composition of $\mathcal{A}$ with $\hat{\mathcal{A}}_0$, is identical to the action of $\mathcal{A}$ (3.3) except that it distinguishes (and interchanges) the topological generators of the two twisted theories, *i.e.* it exchanges the labels (1) and (2). Thus one can substitute $\mathcal{A}$ by $\hat{\mathcal{U}}_1$ or $\hat{\mathcal{U}}_{-1}$ in diagram (4.1). This amounts to differentiate the upper part and the lower part of the diagram as corresponding to topological singular vectors and generators labelled by (1) or by (2), respectively. For example

$$\begin{array}{ccc} |\chi^{(1)}\rangle^{(0)G}_{l,\mathsf{h}} & \xrightarrow{\mathcal{Q}_0^{(1)}} & |\chi^{(1)}\rangle^{(-1)Q}_{l,\mathsf{h}} \\ \hat{\mathcal{U}}_1 \uparrow & & \uparrow \hat{\mathcal{U}}_1 \\ |\chi^{(2)}\rangle^{(0)Q}_{l,-\mathsf{h}-\frac{\mathsf{c}}{3}} & \xrightarrow{\mathcal{G}_0^{(2)}} & |\chi^{(2)}\rangle^{(1)G}_{l,-\mathsf{h}-\frac{\mathsf{c}}{3}} \end{array} \tag{4.2}$$

(one can reverse the arrows $\hat{\mathcal{U}}_1$ using $\hat{\mathcal{U}}_{-1}$). Therefore, the topological singular vectors in chiral Verma modules come actually in families of four plus four vectors, four of them labelled by (1) and the other four identical vectors labelled by (2), the two sets being connected through the action of $\hat{\mathcal{U}}_{\pm 1}$. From the point of view of the Topological algebra (2.1) the two sets of singular vectors and generators look identical, so that one can consider a unique set of four singular vectors, as we did before. However, the topological generators (1) and (2) are different with respect to the generators of the NS algebra. As a consequence diagrams (4.1) and (4.2) give different results under the (un)twistings T_{W1} (2.2) and T_{W2} (2.3).

Regarding the untwisting of these topological singular vectors, the $\mathcal{G}_0$-closed singular vectors in chiral Verma modules are transformed into singular vectors of the NS algebra, with the same U(1) charge and built on antichiral primaries under T_{W1} (2.2), and with the reverse U(1) charge and built on chiral primaries under T_{W2} (2.3) (whereas the $\mathcal{Q}_0$-closed singular vectors are transformed into states which are not NS singular vectors). As a consequence, the fact that the four types of singular vectors in diagram (4.1) are the only existing ones in chiral Verma modules, implies [24] that the singular vectors of the NS algebra built on chiral primaries have only relative charges $q = 0$ or $q = -1$, while those built on antichiral primaries have $q = 0$ or $q = 1$.

Finally, let us point out a result for the NS algebra which is easier to obtain from the Topological algebra. It is the fact that there are no chiral NS singular vectors in chiral Verma modules neither antichiral NS singular vectors in antichiral Verma modules. To see this one only needs to "untwist" the fact that there are no chiral topological singular

vectors in chiral topological Verma modules.

4.2 Spectrum of h

Let us discuss now the spectrum of U(1) charges h corresponding to the topological chiral primaries $|0,\mathsf{h}\rangle^{G,Q}$ which contain singular vectors in their Verma modules. This spectrum follows directly from the spectrum corresponding to the singular vectors of the NS algebra on antichiral Verma modules. Namely, the $\mathcal{G}_0$-closed topological singular vectors on chiral topological primaries become NS singular vectors on antichiral primaries with the same U(1) charge, under T_{W1} (2.2).

Hence the spectrum of U(1) charges h corresponding to the topological chiral Verma modules which contain $\mathcal{G}_0$-closed singular vectors is identical to the spectrum of U(1) charges h corresponding to the antichiral Verma modules of the NS algebra which contain singular vectors. This spectrum has been conjectured in [24] by analyzing the roots of the determinant formula for the NS algebra in a rather non-trivial way (because the determinant formula does not apply directly to chiral or antichiral Verma modules, but only to complete Verma modules).

The results are as follows. A topological chiral primary with U(1) charge given by

$$\mathbf{h}^{(0)}_{r,s} = \frac{3-\mathbf{c}}{6}(r+1) - \frac{s}{2}\ , \qquad r \in \mathbf{Z}^+, \quad s \in 2\mathbf{Z}^+ \tag{4.3}$$

has one singular vector of type $|\chi\rangle^{(0)G}$, and one singular vector of type $|\chi\rangle^{(-1)Q}$, at level $\frac{rs}{2}$ in its Verma module (and possibly more singular vectors at higher levels) . Similarly, a topological chiral primary with U(1) charge given by

$$\mathbf{h}^{(1)}_{r,s} = \frac{\mathbf{c}-3}{6}(r-1) + \frac{s}{2} - 1\ , \qquad r \in \mathbf{Z}^+, \quad s \in 2\mathbf{Z}^+ \tag{4.4}$$

has one singular vector of type $|\chi\rangle^{(1)G}$, and one singular vector of type $|\chi\rangle^{(0)Q}$, at level $\frac{rs}{2}$ in its Verma module[§] (and possibly more singular vectors at higher levels).

These expressions have been checked until level 4 by explicit construction of the singular vectors and by computing the chiral determinant formulae [24]. Observe that between both expressions there exists the spectral flow relation $\mathbf{h}^{(1)}_{rs} = -\mathbf{h}^{(0)}_{rs} - \frac{\mathbf{c}}{3}$. Therefore the two expressions coincide for the special case $\mathbf{h}^{(1)}_{rs} = \mathbf{h}^{(0)}_{rs} = -\frac{\mathbf{c}}{6}$. The solutions to this give discrete values of $\mathbf{c} < 3$. Namely

[§]The spectrum (4.4) for the uncharged $\mathcal{Q}_0$-closed topological singular vectors $|\chi\rangle^{(0)Q}$ was also written down in [16] just by fitting the known data (until level 4), without any derivation or further analysis.

$$\mathbf{c} = \frac{3(r-s+1)}{r}\,, \qquad r \in \mathbf{Z}^+, \quad s \in 2\mathbf{Z}^+ \tag{4.5}$$

For example, for $s = 2$ the solutions corresponding to $r = 1, 2, 3$ are $\mathbf{c} = 0, 3/2, 2$ respectively. Therefore for the discrete values (4.5) the topological chiral Verma modules with $\mathbf{h} = -\frac{\mathbf{c}}{6}$ have four singular vectors, one of each type, at the same level $l = \frac{rs}{2}$.

4.3 Subsingular vectors

Subsingular vectors of the N=2 Superconformal algebras have been recently discovered in ref. [24]. For the case of the Topological algebra we have found that most of the singular vectors given by the spectrum (4.4), are subsingular vectors in the complete Verma modules $V(|0,\mathbf{h}\rangle^G)$ and most of the singular vectors given by the spectrum (4.3), are subsingular vectors in the complete Verma modules $V(|0,\mathbf{h}\rangle^Q)$. The argument goes as follows (all the statements about spectra in complete Verma modules can be checked in subsection 5.2).

The spectrum $\mathbf{h}^{(1)}_{r,s}$ (4.4), corresponding to singular vectors of types $|\chi\rangle^{(1)G}_l$ and $|\chi\rangle^{(0)Q}_l$ in chiral Verma modules, is the "fusion" of two different spectra, $\mathbf{h}_k$ and $\hat{\mathbf{h}}_{r,s}$, given by

$$\mathbf{h}_k = \frac{\mathbf{c}-3}{6}(k-1)\,, \qquad \hat{\mathbf{h}}_{r,s} = \frac{\mathbf{c}-3}{6}(r-1) + \frac{s}{2}\,. \tag{4.6}$$

The spectrum $\mathbf{h}_k$ corresponds to $\mathbf{h}^{(1)}_{r,s}$ for $s = 2$: $\mathbf{h}_k = \mathbf{h}^{(1)}_{k,2}$, with the level of the singular vector given by $l = k$. It coincides with the spectrum of singular vectors of types $|\chi\rangle^{(1)G}_{l,|0,\mathbf{h}\rangle^G}$ and $|\chi\rangle^{(0)Q}_{l,|0,\mathbf{h}\rangle^G}$ in the complete Verma modules $V(|0,\mathbf{h}\rangle^G)$. The spectrum $\hat{\mathbf{h}}_{r,s}$, on the other hand, gives $\mathbf{h}^{(1)}_{r,s}$ for $s > 2$ in the way: $\hat{\mathbf{h}}_{r,s-2} = \mathbf{h}^{(1)}_{r,s>2}$, where the level of the singular vector is given by $l = rs/2$. It turns out that $\hat{\mathbf{h}}_{r,s}$ corresponds to half the spectrum of singular vectors of types $|\chi\rangle^{(0)G}_{l,|0,\mathbf{h}\rangle^G}$ and $|\chi\rangle^{(-1)Q}_{l,|0,\mathbf{h}\rangle^G}$ in the complete Verma modules $V(|0,\mathbf{h}\rangle^G)$ (the other half being given by (4.3)), with the level given by $l = rs/2$. Therefore $\hat{\mathbf{h}}_{r,s-2} = \mathbf{h}^{(1)}_{r,s>2}$ corresponds either to singular vectors of types $|\chi\rangle^{(1)G}_l$ and $|\chi\rangle^{(0)Q}_l$ in chiral Verma modules $V(|0,\mathbf{h}\rangle^{G,Q})$, at level $l = rs/2$, or to singular vectors of types $|\chi\rangle^{(0)G}_{l',|0,\mathbf{h}\rangle^G}$ and $|\chi\rangle^{(-1)Q}_{l',|0,\mathbf{h}\rangle^G}$ in complete Verma modules $V(|0,\mathbf{h}\rangle^G)$ at level $l' = r(s-2)/2$.

It happens, at least at levels 2 and 3, that the latter singular vectors vanish, while the former ones appear from secondaries which become singular, once one imposes the chirality condition $\mathcal{Q}_0|0,\mathbf{h}\rangle^G = 0$ on the primary $|0,\mathbf{h}\rangle^G$, turning it into the chiral primary $|0,\mathbf{h}\rangle^{G,Q}$. But $\mathcal{Q}_0|0,\mathbf{h}\rangle^G$ is a singular vector so that the chirality condition is equivalent to take the quotient of the complete Verma module $V(|0,\mathbf{h}\rangle^G)$ by this singular vector.

Those secondaries are therefore subsingular vectors [28], since they are singular only in the chiral Verma module, that is, in the quotient of the Verma module $V(|0,\mathbf{h}\rangle^G)$ by the submodule generated by the singular vector $\mathcal{Q}_0|0,\mathbf{h}\rangle^G$.

Hence the singular vectors of types $|\chi\rangle_l^{(1)G}$ and $|\chi\rangle_l^{(0)Q}$ in chiral Verma modules are subsingular vectors in the complete Verma modules $V(|0,\mathbf{h}\rangle^G)$, for the values $\mathbf{h}^{(1)}_{r,s>2}$. The subsingular vectors are null and, however, are located outside the (incomplete) Verma modules built on top of the singular vectors.

Similarly, the spectrum $\mathbf{h}^{(0)}_{r,s}$ (4.3), corresponding to singular vectors of types $|\chi\rangle_l^{(0)G}$ and $|\chi\rangle_l^{(-1)Q}$ in chiral Verma modules, is the fusion of two different spectra: $\mathbf{h}^{(0)}_{r,2}$ which corresponds to singular vectors of types $|\chi\rangle^{(0)G}_{l,|0,\mathbf{h}\rangle^Q}$ and $|\chi\rangle^{(-1)Q}_{l,|0,\mathbf{h}\rangle^Q}$, and $\mathbf{h}^{(0)}_{r,s>2}$ which, together with $\mathbf{h}^{(1)}_{r,s}$, corresponds to singular vectors of types $|\chi\rangle^{(0)Q}_{l,|0,\mathbf{h}\rangle^Q}$ and $|\chi\rangle^{(1)G}_{l,|0,\mathbf{h}\rangle^Q}$, all these vectors in the complete Verma modules $V(|0,\mathbf{h}\rangle^Q)$. Therefore, the singular vectors of types $|\chi\rangle_l^{(0)G}$ and $|\chi\rangle_l^{(-1)Q}$ in chiral Verma modules are subsingular vectors in the complete Verma modules $V(|0,\mathbf{h}\rangle^Q)$ for the values $\mathbf{h}^{(0)}_{r,s>2}$; that is, they are not singular vectors in $V(|0,\mathbf{h}\rangle^Q)$ but they are singular vectors in the quotient of $V(|0,\mathbf{h}\rangle^Q)$ by the submodule generated by the singular vector $\mathcal{G}_0|0,\mathbf{h}\rangle^Q$.

These results we have checked at levels 2 and 3 and we conjecture that they hold at any level. In the next subsection we show the families of singular vectors in chiral Verma modules until level 3 and we identify the subsingular vectors contained in them.

4.4 Families at Levels 1, 2 and 3

Let us write down the complete families of topological singular vectors in chiral Verma modules until level 3. Some of these vectors have been published before: $|\chi\rangle_2^{(0)Q}$ and $|\chi\rangle_3^{(0)Q}$ were written in [14] and [15] respectively, while the complete family at level 2 was given in [18]. The fact that most of these singular vectors are subsingular vectors in the complete Verma modules $V(|0,\mathbf{h}\rangle^G)$ or $V(|0,\mathbf{h}\rangle^Q)$ was not realized, however.

The families of topological singular vectors in chiral Verma modules are the following.

At level 1:

$$|\chi\rangle_1^{(0)G} = (\mathcal{L}_{-1} + \mathcal{H}_{-1})|0, -\frac{\mathbf{c}}{3}\rangle^{G,Q}, \qquad |\chi\rangle_1^{(0)Q} = \mathcal{L}_{-1}|0,0\rangle^{G,Q}, \tag{4.7}$$

$$|\chi\rangle_1^{(-1)Q} = \mathcal{Q}_{-1}|0, -\frac{\mathbf{c}}{3}\rangle^{G,Q}, \qquad |\chi\rangle_1^{(1)G} = \mathcal{G}_{-1}|0,0\rangle^{G,Q}. \tag{4.8}$$

At level 2:

$$|\chi\rangle_2^{(0)G} = (\theta\mathcal{L}_{-2} + \alpha\mathcal{L}_{-1}^2 + \Gamma\mathcal{H}_{-1}\mathcal{L}_{-1} + \beta\mathcal{H}_{-1}^2 + \gamma\mathcal{H}_{-2} + \delta\mathcal{Q}_{-1}\mathcal{G}_{-1})|0,\mathbf{h}\rangle^{G,Q} \tag{4.9}$$

$$\mathbf{h} = \begin{cases} \frac{1-\mathbf{c}}{2} \\ -\frac{\mathbf{c}+3}{3} \end{cases}, \qquad \alpha = \begin{cases} \frac{6}{\mathbf{c}-3} \\ \frac{\mathbf{c}-3}{6} \end{cases}, \qquad \theta = \begin{cases} \frac{9-\mathbf{c}}{3-\mathbf{c}} \\ 0 \end{cases}, \qquad \Gamma = \begin{cases} \frac{18}{\mathbf{c}-3} \\ \frac{\mathbf{c}}{3} \end{cases}$$
$$\beta = \begin{cases} \frac{12}{\mathbf{c}-3} \\ \frac{\mathbf{c}+3}{6} \end{cases}, \qquad \gamma = \begin{cases} 2 \\ \frac{\mathbf{c}+3}{6} \end{cases}, \qquad \delta = \begin{cases} \frac{3}{\mathbf{c}-3} \\ \frac{1}{2} \end{cases} \tag{4.10}$$

$$|\chi\rangle_2^{(0)Q} = (\mathcal{L}_{-2} + \alpha\mathcal{L}_{-1}^2 + \Gamma\mathcal{H}_{-1}\mathcal{L}_{-1} + \frac{1}{2}\Gamma\mathcal{Q}_{-1}\mathcal{G}_{-1})|0,\mathbf{h}\rangle^{G,Q} \tag{4.11}$$

$$\mathbf{h} = \begin{cases} \frac{\mathbf{c}-3}{6} \\ 1 \end{cases}, \qquad \alpha = \begin{cases} \frac{6}{\mathbf{c}-3} \\ \frac{\mathbf{c}-3}{6} \end{cases}, \qquad \Gamma = \begin{cases} \frac{6}{3-\mathbf{c}} \\ -1 \end{cases} \tag{4.12}$$

$$|\chi\rangle_2^{(1)G} = (\mathcal{G}_{-2} + \alpha\mathcal{L}_{-1}\mathcal{G}_{-1} + \Gamma\mathcal{H}_{-1}\mathcal{G}_{-1})|0,\mathbf{h}\rangle^{G,Q} \tag{4.13}$$

$$\mathbf{h} = \begin{cases} \frac{\mathbf{c}-3}{6} \\ 1 \end{cases}, \qquad \alpha = \begin{cases} \frac{6}{\mathbf{c}-3} \\ \frac{\mathbf{c}-3}{6} \end{cases}, \qquad \Gamma = \begin{cases} \frac{6}{3-\mathbf{c}} \\ -1 \end{cases} \tag{4.14}$$

$$|\chi\rangle_2^{(-1)Q} = (\mathcal{Q}_{-2} + \alpha\mathcal{L}_{-1}\mathcal{Q}_{-1} + \beta\mathcal{H}_{-1}\mathcal{Q}_{-1})|0,\mathbf{h}\rangle^{G,Q} \tag{4.15}$$

$$\mathbf{h} = \begin{cases} \frac{1-\mathbf{c}}{2} \\ -\frac{\mathbf{c}+3}{3} \end{cases}, \qquad \alpha = \begin{cases} \frac{6}{\mathbf{c}-3} \\ \frac{\mathbf{c}-3}{6} \end{cases}, \qquad \beta = \begin{cases} \frac{12}{\mathbf{c}-3} \\ \frac{\mathbf{c}+3}{6} \end{cases} \tag{4.16}$$

At level 3:

$$\begin{aligned} |\chi\rangle_3^{(0)G} = (&\alpha\mathcal{L}_{-1}^3 + \theta\mathcal{L}_{-2}\mathcal{L}_{-1} + \beta\mathcal{H}_{-3} + \gamma\mathcal{H}_{-2}\mathcal{L}_{-1} + \delta\mathcal{L}_{-3} + \\ &\epsilon\mathcal{H}_{-1}\mathcal{L}_{-2} + \mu\mathcal{H}_{-1}^2\mathcal{L}_{-1} + \nu\mathcal{H}_{-1}\mathcal{L}_{-1}^2 + \kappa\mathcal{H}_{-1}\mathcal{H}_{-2} + \rho\mathcal{H}_{-1}^3 + \\ &a\mathcal{Q}_{-2}\mathcal{G}_{-1} + e\mathcal{Q}_{-1}\mathcal{G}_{-2} + f\mathcal{L}_{-1}\mathcal{Q}_{-1}\mathcal{G}_{-1} + g\mathcal{H}_{-1}\mathcal{Q}_{-1}\mathcal{G}_{-1})|0,\mathbf{h}\rangle^{G,Q} \end{aligned}$$

$$\mathbf{h} = \begin{cases} \frac{3-2\mathbf{c}}{3} \\ -\frac{\mathbf{c}+6}{3} \end{cases}, \qquad \alpha = \begin{cases} \frac{3}{3-\mathbf{c}} \\ \frac{3-\mathbf{c}}{12} \end{cases}, \qquad \theta = \begin{cases} \frac{15-2\mathbf{c}}{\mathbf{c}-3} \\ -\frac{1}{2} \end{cases}, \qquad \beta = \begin{cases} 3-\mathbf{c} \\ \frac{\mathbf{c}^2+12\mathbf{c}+27}{6(3-\mathbf{c})} \end{cases}$$

$$\delta = \begin{cases} \frac{(\mathbf{c}-6)^2}{3(3-\mathbf{c})} \\ -\frac{1}{2} \end{cases}, \qquad \gamma = \begin{cases} \frac{5\mathbf{c}-12}{3-\mathbf{c}} \\ \frac{\mathbf{c}^2+6\mathbf{c}-3}{4(3-\mathbf{c})} \end{cases}, \qquad \epsilon = \begin{cases} \frac{33-4\mathbf{c}}{\mathbf{c}-3} \\ -\frac{1}{2} \end{cases}, \qquad \nu = \begin{cases} \frac{18}{3-\mathbf{c}} \\ -\frac{\mathbf{c}+3}{4} \end{cases}$$

$$\mu = \begin{cases} \frac{33}{3-\mathbf{c}} \\ \frac{\mathbf{c}^2+6\mathbf{c}-3}{4(3-\mathbf{c})} \end{cases}, \qquad a = \begin{cases} -1 \\ \frac{6}{3-\mathbf{c}} \end{cases}, \qquad e = \begin{cases} \frac{1}{2}\frac{15-\mathbf{c}}{\mathbf{c}-3} \\ \frac{1}{2} \end{cases}, \qquad \kappa = \begin{cases} -9 \\ \frac{\mathbf{c}^2+12\mathbf{c}+27}{4(3-\mathbf{c})} \end{cases}$$

$$\rho = \begin{cases} \frac{18}{3-\mathbf{c}} \\ \frac{\mathbf{c}^2+12\mathbf{c}+27}{12(3-\mathbf{c})} \end{cases}, \qquad f = \begin{cases} \frac{9}{2(3-\mathbf{c})} \\ -\frac{3}{4} \end{cases}, \qquad g = \begin{cases} \frac{21}{2(3-\mathbf{c})} \\ \frac{3\mathbf{c}+15}{4(3-\mathbf{c})} \end{cases} \tag{4.17}$$

$$|\chi\rangle_3^{(0)Q} = (\alpha\mathcal{L}_{-1}^3 - 2\mathcal{L}_{-2}\mathcal{L}_{-1} + \gamma\mathcal{H}_{-2}\mathcal{L}_{-1} + \delta\mathcal{L}_{-3} + 2e\mathcal{H}_{-1}\mathcal{L}_{-2} + g\mathcal{H}_{-1}^2\mathcal{L}_{-1}$$
$$+2f\mathcal{H}_{-1}\mathcal{L}_{-1}^2 + a\mathcal{Q}_{-2}\mathcal{G}_{-1} + e\mathcal{Q}_{-1}\mathcal{G}_{-2} + f\mathcal{L}_{-1}\mathcal{Q}_{-1}\mathcal{G}_{-1} + g\mathcal{H}_{-1}\mathcal{Q}_{-1}\mathcal{G}_{-1})|0,\mathbf{h}\rangle^{G,Q}$$

$$\mathbf{h} = \begin{cases} \frac{\mathbf{c}-3}{3} \\ 2 \end{cases}, \qquad \alpha = \begin{cases} \frac{3}{3-\mathbf{c}} \\ \frac{3-\mathbf{c}}{12} \end{cases}, \qquad \gamma = \begin{cases} \frac{\mathbf{c}}{\mathbf{c}-3} \\ \frac{1}{2}\frac{\mathbf{c}+9}{\mathbf{c}-3} \end{cases}, \qquad \delta = \begin{cases} -\frac{\mathbf{c}}{3} \\ \frac{9+\mathbf{c}}{3-\mathbf{c}} \end{cases}$$

$$a = \begin{cases} \frac{1}{2}\frac{15-\mathbf{c}}{3-\mathbf{c}} \\ -\frac{1}{2} \end{cases}, \qquad e = \begin{cases} 1 \\ \frac{6}{\mathbf{c}-3} \end{cases}, \qquad f = \begin{cases} \frac{1}{2}\frac{9}{\mathbf{c}-3} \\ \frac{3}{4} \end{cases}, \qquad g = \begin{cases} \frac{6}{3-\mathbf{c}} \\ \frac{6}{3-\mathbf{c}} \end{cases} \tag{4.18}$$

$$|\chi\rangle_3^{(1)G} = (\alpha\mathcal{L}_{-1}^2\mathcal{G}_{-1} + \beta\mathcal{L}_{-1}\mathcal{G}_{-2} + \epsilon\mathcal{L}_{-2}\mathcal{G}_{-1} + \gamma\mathcal{H}_{-2}\mathcal{G}_{-1} + \theta\mathcal{G}_{-3} +$$
$$e\mathcal{H}_{-1}\mathcal{G}_{-2} + f\mathcal{H}_{-1}\mathcal{L}_{-1}\mathcal{G}_{-1} + g\mathcal{H}_{-1}^2\mathcal{G}_{-1})|0,\mathbf{h}\rangle^{G,Q}$$

$$\mathbf{h} = \begin{cases} \frac{\mathbf{c}-3}{3} \\ 2 \end{cases}, \qquad \alpha = \begin{cases} \frac{3}{2(3-\mathbf{c})} \\ \frac{3-\mathbf{c}}{24} \end{cases}, \qquad \beta = \begin{cases} \frac{\mathbf{c}}{2(3-\mathbf{c})} \\ -\frac{3}{4} \end{cases}$$

$$\epsilon = \begin{cases} \frac{\mathbf{c}-6}{2(3-\mathbf{c})} \\ -\frac{1}{4} \end{cases}, \qquad \gamma = \begin{cases} \frac{\mathbf{c}}{2(\mathbf{c}-3)} \\ \frac{\mathbf{c}+9}{4(\mathbf{c}-3)} \end{cases}, \qquad \theta = \begin{cases} \frac{\mathbf{c}}{6}\frac{\mathbf{c}-6}{3-\mathbf{c}} \\ \frac{27-\mathbf{c}}{4(3-\mathbf{c})} \end{cases}$$

$$e = \begin{cases} 1 \\ \frac{6}{\mathbf{c}-3} \end{cases}, \qquad f = \begin{cases} \frac{9}{2(\mathbf{c}-3)} \\ \frac{3}{4} \end{cases}, \qquad g = \begin{cases} \frac{3}{3-\mathbf{c}} \\ \frac{3}{3-\mathbf{c}} \end{cases} \tag{4.19}$$

$$|\chi\rangle_3^{(-1)Q} = (\alpha\mathcal{L}_{-1}^2\mathcal{Q}_{-1} + \beta\mathcal{L}_{-1}\mathcal{Q}_{-2} + \epsilon\mathcal{L}_{-2}\mathcal{Q}_{-1} + \gamma\mathcal{H}_{-2}\mathcal{Q}_{-1}$$
$$+\theta\mathcal{Q}_{-3} + e\mathcal{H}_{-1}\mathcal{Q}_{-2} + f\mathcal{H}_{-1}\mathcal{L}_{-1}\mathcal{Q}_{-1} + g\mathcal{H}_{-1}^2\mathcal{Q}_{-1})|0,\mathbf{h}\rangle^{G,Q}$$

$$\mathbf{h} = \begin{cases} \frac{3-2\mathbf{c}}{3} \\ -\frac{\mathbf{c}+6}{3} \end{cases}, \qquad \alpha = \begin{cases} \frac{9}{3-\mathbf{c}} \\ \frac{3-\mathbf{c}}{4} \end{cases}, \qquad \beta = \begin{cases} \frac{3\mathbf{c}}{3-\mathbf{c}} \\ -\frac{9}{2} \end{cases}$$

$$\epsilon = \begin{cases} \frac{18-3\mathbf{c}}{\mathbf{c}-3} \\ -\frac{3}{2} \end{cases}, \qquad \gamma = \begin{cases} -9 \\ \frac{\mathbf{c}^2+12\mathbf{c}+27}{4(3-\mathbf{c})} \end{cases}, \qquad \theta = \begin{cases} \frac{\mathbf{c}^2-6\mathbf{c}}{3-\mathbf{c}} \\ \frac{81-3\mathbf{c}}{2(3-\mathbf{c})} \end{cases}$$

$$e = \begin{cases} \frac{9\mathbf{c}-18}{3-\mathbf{c}} \\ \frac{9\mathbf{c}+45}{2(3-\mathbf{c})} \end{cases}, \qquad f = \begin{cases} \frac{45}{3-\mathbf{c}} \\ -\frac{\mathbf{c}+6}{2} \end{cases}, \qquad g = \begin{cases} \frac{54}{3-\mathbf{c}} \\ \frac{\mathbf{c}^2+12\mathbf{c}+27}{4(3-\mathbf{c})} \end{cases} \tag{4.20}$$

At level 1 there are no subsingular vectors because $(r,s)=(1,2)$, *i.e.* the only possible value of s is 2. At level 2 the singular vectors $|\chi\rangle_2^{(1)G}$ and $|\chi\rangle_2^{(0)Q}$, for $\mathbf{h}_{1,4}^{(1)}=1$, are subsingular in the complete Verma module $V(|0,1\rangle^G)$. That is, $|\hat{\chi}\rangle_{2,|0,1\rangle^G}^{(1)G}$ and $|\hat{\chi}\rangle_{2,|0,1\rangle^G}^{(0)\hat{Q}}$ given by

$$|\hat{\chi}\rangle_{2,|0,1\rangle^G}^{(1)G} = (\mathcal{G}_{-2} + \frac{\mathbf{c}-3}{6}\mathcal{L}_{-1}\mathcal{G}_{-1} - \mathcal{H}_{-1}\mathcal{G}_{-1})|0,1\rangle^G \tag{4.21}$$

and

$$|\hat{\chi}\rangle_{2,|0,1\rangle^G}^{(0)\hat{Q}} = (\mathcal{L}_{-2} + \frac{\mathbf{c}-3}{6}\mathcal{L}_{-1}^2 - \mathcal{H}_{-1}\mathcal{L}_{-1} - \frac{1}{2}\mathcal{Q}_{-1}\mathcal{G}_{-1})|0,1\rangle^G \tag{4.22}$$

are subsingular vectors (the hat denotes that they are not h.w. vectors while $\hat{Q}$ denotes that the vector is not $\mathcal{Q}_0$-closed anymore). The positive mode $\mathcal{Q}_1$ brings $|\hat{\chi}\rangle_{2,|0,1\rangle^G}^{(1)G}$ "down" to the singular vector $|\chi\rangle_{1,|0,1\rangle^G}^{(0)G} = \mathcal{G}_{-1}\mathcal{Q}_0|0,1\rangle^G$ (for $\mathbf{c}\neq 9$ since $\mathcal{Q}_1|\hat{\chi}\rangle_{2,|0,1\rangle^G}^{(1)G} = \frac{(9-\mathbf{c})}{6}\mathcal{G}_{-1}\mathcal{Q}_0|0,1\rangle^G$), which is a secondary singular vector built on the level-zero singular vector $|\chi\rangle_{0,|0,1\rangle^G}^{(-1)G,Q} = \mathcal{Q}_0|0,1\rangle^G$. Similarly, the positive modes $\mathcal{L}_1$ and $\mathcal{H}_1$ bring $|\hat{\chi}\rangle_{2,|0,1\rangle^G}^{(0)\hat{Q}}$ down to the singular vector $|\chi\rangle_{1,|0,1\rangle^G}^{(0)G}$. From this one it is not possible to reach the subsingular vectors acting with the negative modes, however. That is, the subsingular vectors sit outside the (incomplete) Verma modules built on top of the singular vectors $|\chi\rangle_{1,|0,1\rangle^G}^{(0)G}$ and $|\chi\rangle_{0,|0,1\rangle^G}^{(-1)G,Q}$. This is shown in Figure 1.

Similarly, the singular vectors $|\chi\rangle_2^{(0)G}$ and $|\chi\rangle_2^{(-1)Q}$, for $\mathbf{h}_{1,4}^{(0)} = -\frac{\mathbf{c}+3}{3}$, are subsingular in the complete Verma module $V(|0,-\frac{\mathbf{c}+3}{3}\rangle^Q)$. Both of them descend to the singular vector $|\chi\rangle_{1,|0,-\frac{\mathbf{c}+3}{3}\rangle^Q}^{(0)Q} = \mathcal{Q}_{-1}\mathcal{G}_0|0,-\frac{\mathbf{c}+3}{3}\rangle^Q$, for $\mathbf{c}\neq 9$, which is a secondary singular vector built on the level-zero singular vector $|\chi\rangle_{0,|0,-\frac{\mathbf{c}+3}{3}\rangle^Q}^{(1)G,Q} = \mathcal{G}_0|0,-\frac{\mathbf{c}+3}{3}\rangle^Q$.

At level 3 the singular vectors $|\chi\rangle_3^{(1)G}$ and $|\chi\rangle_3^{(0)Q}$, for $\mathbf{h}_{1,6}^{(1)}=2$, are subsingular in the complete Verma module $V(|0,2\rangle^G)$ whereas the singular vectors $|\chi\rangle_3^{(0)G}$ and $|\chi\rangle_3^{(-1)Q}$, for $\mathbf{h}_{1,6}^{(0)} = -\frac{\mathbf{c}+6}{3}$, are subsingular in the complete Verma module $V(|0,-\frac{\mathbf{c}+6}{3}\rangle^Q)$. In these

cases the subsingular vectors do not descend to any secondary singular vectors but to the level-zero singular vectors $|\chi\rangle^{(-1)G,Q}_{0,|0,2\rangle^G} = \mathcal{Q}_0|0,2\rangle^G$ and $|\chi\rangle^{(1)G,Q}_{0,|0,-\frac{\mathbf{c}+6}{3}\rangle^Q} = \mathcal{G}_0|0,-\frac{\mathbf{c}+6}{3}\rangle^Q$, for $\mathbf{c} \neq 9$ as before.

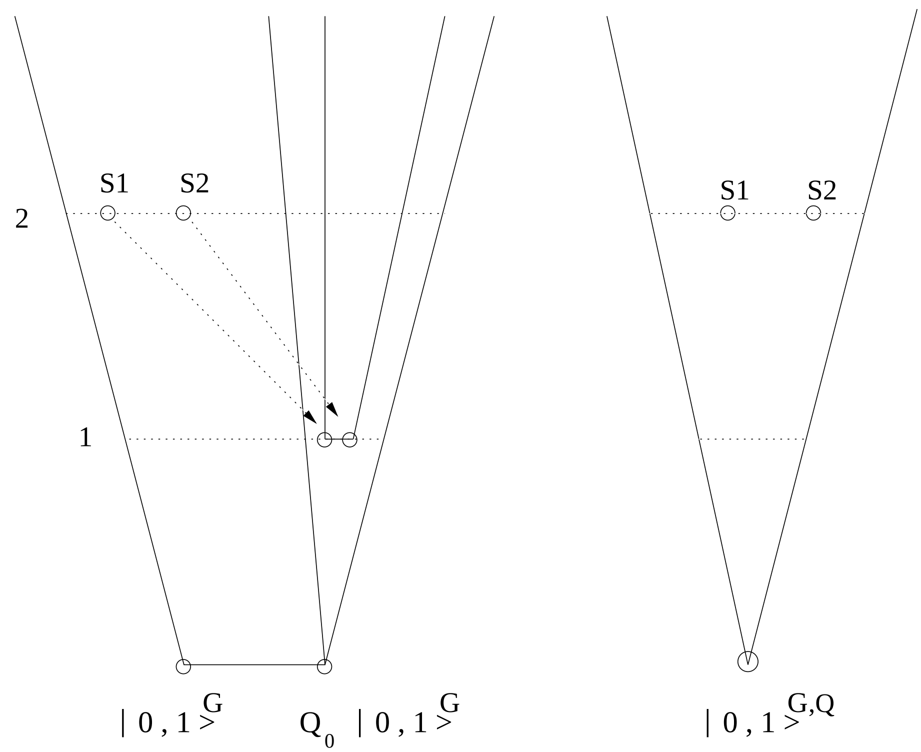

Figure 1: $S1$ and $S2$ represent the subsingular vectors in the complete Verma module $V(|0,1>^G)$, on the left, which become the singular vectors $|\chi\rangle^{(1)G}_2$ and $|\chi\rangle^{(0)Q}_2$ in the chiral Verma module $V(|0,1>^{G,Q})$, on the right. $V(|0,1>^{G,Q})$ is the quotient of $V(|0,1>^G)$ by the submodule generated by the level-zero singular vector $\mathcal{Q}_0|0,1>^G$. The subsingular vectors sit outside the (incomplete) Verma modules built on the singular vectors at levels zero and 1, to which they descend.

5 Families of Topological Singular Vectors in Complete Verma Modules

5.1 Family Structure

Let us apply the results of sections 2 and 3 to derive the family structure of the singular vectors in complete Verma modules. One can expect a much richer structure than in the case of chiral Verma modules since, on the one hand, there are many more types of singular vectors, twenty-nine versus four, and, on the other, there is an additional spectral flow mapping, $\mathcal{U}_{\pm 1}$, which can extend the four-member subfamilies of singular vectors given by the actions of $\mathcal{G}_0$, $\mathcal{Q}_0$ and $\mathcal{A}$ (two-member subfamilies rather if the singular vectors are chiral)[¶]. The label-exchange operator $\hat{\mathcal{A}}_0$ can be composed with $\mathcal{A}$ and $\mathcal{U}_{\pm 1}$ to produce the same transformations as these ones but connecting the two sets of topological generators and states labeled by (1) and (2). In the diagrams presented below we will not consider $\hat{\mathcal{A}}_0$, for simplicity, although this possibility must be taken into account, as we did in the case of chiral Verma modules in last section (diagram (4.2)).

Let us start with the generic Verma modules $V(|\Delta, \mathbf{h}\rangle^G)$ and $V(|\Delta, \mathbf{h}\rangle^Q)$. The types of singular vectors allowed by the cascade effect are twenty, as shown in tables (2.6) and (2.7): twelve types of generic singular vectors (six $\mathcal{G}_0$-closed and six $\mathcal{Q}_0$-closed), plus four types of chiral singular vectors, plus four types of no-label singular vectors. The twelve types of generic singular vectors and the four types of chiral singular vectors can be mapped to singular vectors of the NS algebra, as we will see. Therefore one can write down general construction formulae for them, using the construction formulae for the NS singular vectors [8] [9]. The four types of no-label singular vectors, however, are not related to NS singular vectors. These twenty types of singular vectors exist already at level 1 (see Appendix B).

First kind of generic families of topological singular vectors

Let us analyze how the generic and chiral types of singular vectors are organized into families. The key fact is that using $\mathcal{U}_{\pm 1}$ one can extend a topological subfamily of four (or two) singular vectors given by the actions of $\mathcal{G}_0$, $\mathcal{Q}_0$ and $\mathcal{A}$, resulting in, at least, two subfamilies located in four different Verma modules, two each. These two subfamilies will be denoted as "the skeleton-family".

Let us start with an uncharged singular vector of type $|\chi\rangle^{(0)G}_{l,|\phi\rangle^G}$ in the Verma module $V(|\Delta, \mathbf{h}\rangle^G)$. As diagram (5.1) shows, the members of its skeleton-family are, in the most general case, singular vectors of the types $|\chi\rangle^{(0)G}_{l,|\phi\rangle^G}$ and $|\chi\rangle^{(-1)Q}_{l,|\phi\rangle^G}$, in both the Verma modules $V(|\Delta, \mathbf{h}\rangle^G)$ and $V(|\Delta - \mathbf{h}, -\mathbf{h}\rangle^G)$, and singular vectors of the types $|\chi\rangle^{(0)Q}_{l,|\phi\rangle^Q}$ and $|\chi\rangle^{(1)G}_{l,|\phi\rangle^Q}$, in both the Verma modules $V(|\Delta - \mathbf{h}, \mathbf{h} - \mathbf{c}/3\rangle^Q)$ and $V(|\Delta, -\mathbf{h} - \mathbf{c}/3\rangle^Q)$.

In the special case $\Delta = -l$ the conformal weight of the singular vectors in the lower subdiagram is zero, so that the corresponding arrows $\mathcal{Q}_0$, $\mathcal{G}_0$ cannot be reversed, producing secondary chiral singular vectors $|\chi\rangle^{(-1)G,Q}_{l,|\Delta,\mathbf{h}\rangle^G}$ and $|\chi\rangle^{(1)G,Q}_{l,|\Delta,-\mathbf{h}-\mathbf{c}/3\rangle^Q}$ on the right-

[¶] We remind the reader that chiral singular vectors $|\chi\rangle^{(q)G,Q}_{l}$, *i.e.* annihilated by both $\mathcal{G}_0$ and $\mathcal{Q}_0$, do not exist in chiral Verma modules, they exist in generic Verma modules $V(|\Delta, \mathbf{h}\rangle^G)$ and $V(|\Delta, \mathbf{h}\rangle^Q)$ with $l = -\Delta$, and at level zero also in no-label Verma modules $V(|0, \mathbf{h}\rangle)$: $|\chi\rangle^{(0)G,Q}_{0,|0,\mathbf{h}\rangle} = \mathcal{Q}_0\mathcal{G}_0|0, \mathbf{h}\rangle$.

hand side, at level zero with respect to the singular vectors on the left-hand side.

In the special case $\Delta - \mathbf{h} = -l$ the conformal weight of the singular vectors in the upper subdiagram is zero so that one of the following two possibilities must happen:

a) The singular vectors denoted by $|\chi\rangle^{(0)G}_{l,|\Delta-\mathbf{h},-\mathbf{h}\rangle^G}$ and $|\chi\rangle^{(0)Q}_{l,|\Delta-\mathbf{h},\,\mathbf{h}-\mathbf{c}/3\rangle^Q}$ turn out to be chiral, *i.e.* of types $|\chi\rangle^{(0)G,Q}_{l,|\Delta-\mathbf{h},-\mathbf{h}\rangle^G}$ and $|\chi\rangle^{(0)G,Q}_{l,|\Delta-\mathbf{h},\,\mathbf{h}-\mathbf{c}/3\rangle^Q}$ instead, so that the singular vectors $|\chi\rangle^{(-1)Q}_{l,|\Delta-\mathbf{h},-\mathbf{h}\rangle^G}$ and $|\chi\rangle^{(1)G}_{l,|\Delta-\mathbf{h},\,\mathbf{h}-\mathbf{c}/3\rangle^Q}$ are absent.

b) The singular vectors $|\chi\rangle^{(0)G}_{l,|\Delta-\mathbf{h},-\mathbf{h}\rangle^G}$ and $|\chi\rangle^{(0)Q}_{l,|\Delta-\mathbf{h},\,\mathbf{h}-\mathbf{c}/3\rangle^Q}$ are not chiral, therefore the corresponding arrows $\mathcal{Q}_0$, $\mathcal{G}_0$ cannot be reversed producing secondary chiral singular vectors $|\chi\rangle^{(-1)G,Q}_{l,|\Delta-\mathbf{h},-\mathbf{h}\rangle^G}$ and $|\chi\rangle^{(1)G,Q}_{l,|\Delta-\mathbf{h},\,\mathbf{h}-\mathbf{c}/3\rangle^Q}$ at level zero with respect to the singular vectors on the left-hand side.

$$
\begin{array}{ccc}
\mathcal{U}_1 \uparrow & & \\
|\chi\rangle^{(0)G}_{l,|\Delta-\mathbf{h},-\mathbf{h}\rangle^G} & \xrightarrow{\mathcal{Q}_0} & |\chi\rangle^{(-1)Q}_{l,|\Delta-\mathbf{h},-\mathbf{h}\rangle^G} \\
\mathcal{A} \updownarrow & & \updownarrow \mathcal{A} \\
|\chi\rangle^{(0)Q}_{l,|\Delta-\mathbf{h},\,\mathbf{h}-\frac{\mathbf{c}}{3}\rangle^Q} & \xrightarrow{\mathcal{G}_0} & |\chi\rangle^{(1)G}_{l,|\Delta-\mathbf{h},\,\mathbf{h}-\frac{\mathbf{c}}{3}\rangle^Q} \\
\mathcal{U}_1 \uparrow & & \\
|\chi\rangle^{(0)G}_{l,|\Delta,\,\mathbf{h}\rangle^G} & \xrightarrow{\mathcal{Q}_0} & |\chi\rangle^{(-1)Q}_{l,|\Delta,\,\mathbf{h}\rangle^G} \\
\mathcal{A} \updownarrow & & \updownarrow \mathcal{A} \\
|\chi\rangle^{(0)Q}_{l,|\Delta,-\mathbf{h}-\frac{\mathbf{c}}{3}\rangle^Q} & \xrightarrow{\mathcal{G}_0} & |\chi\rangle^{(1)G}_{l,|\Delta,-\mathbf{h}-\frac{\mathbf{c}}{3}\rangle^Q}
\end{array}
\tag{5.1}
$$

The upper row and the lower row of diagram (5.1) are connected by $\mathcal{U}_1$, as indicated by the arrow on top, since $\mathcal{U}_1\mathcal{A}\mathcal{U}_1\mathcal{A} = \mathbf{I}$. It has therefore the topology of a circle. There are no arrows $\mathcal{U}_{\pm1}$ coming in or out of the singular vectors on the right-hand side (unless they turn out to be chiral) because these types do not transform into singular vectors under $\mathcal{U}_{\pm1}$. In the most general case, however, it will be possible to go beyond this limitation and attach more subfamilies to the skeleton-family, as we will see.

If we start with an uncharged chiral singular vector in the Verma module $V(|\Delta,\mathbf{h}\rangle^G)$, with $\Delta = -l$, then the skeleton-family is as in diagram (5.1) but with the lower subdiagram reduced to the couple of chiral singular vectors $|\chi\rangle^{(0)G,Q}_{l,|-l,\,\mathbf{h}\rangle^G}$ and $|\chi\rangle^{(0)G,Q}_{l,|-l,-\mathbf{h}-\mathbf{c}/3\rangle^Q}$. In addition, if $\mathbf{h} = 0$ the singular vectors in the upper subdiagram have zero conformal weight also, so that this subdiagram contains chiral singular vectors as well.

The first kind of generic families contains therefore the four generic types of singular vectors $|\chi\rangle^{(0)G}_{|\phi\rangle^G}$, $|\chi\rangle^{(0)Q}_{|\phi\rangle^Q}$, $|\chi\rangle^{(1)G}_{|\phi\rangle^Q}$ and $|\chi\rangle^{(-1)Q}_{|\phi\rangle^G}$, and the four chiral types of singular vectors

as particular cases, all of them at the same level l. The untwisting of the singular vectors of types $|\chi\rangle^{(0)G}_{|\phi\rangle^G}$ and $|\chi\rangle^{(0)G,Q}_{|\phi\rangle^G}$ produces uncharged singular vectors of the NS algebra. As a result, all the singular vectors of the first kind of generic families can be mapped to uncharged NS singular vectors.

Second kind of generic families of topological singular vectors

Now let us take a charged singular vector of type $|\chi\rangle^{(1)G}_{l,|\phi\rangle^G}$ in the Verma module $V(|\Delta,\mathbf{h}\rangle^G)$. As diagram (5.2) shows, the other members of its skeleton-family are, in the most general case, singular vectors of the types: $|\chi\rangle^{(0)Q}_{l,|\phi\rangle^G}$ in the same Verma module, $|\chi\rangle^{(1)Q}_{l-1,|\phi\rangle^Q}$ and $|\chi\rangle^{(2)G}_{l-1,|\phi\rangle^Q}$ in the Verma module $V(|\Delta-\mathbf{h},\mathbf{h}-\mathbf{c}/3\rangle^Q)$, $|\chi\rangle^{(-1)Q}_{l,|\phi\rangle^Q}$ and $|\chi\rangle^{(0)G}_{l,|\phi\rangle^Q}$ in the Verma module $V(|\Delta,-\mathbf{h}-\mathbf{c}/3\rangle^Q)$, and $|\chi\rangle^{(-1)G}_{l-1,|\phi\rangle^G}$ and $|\chi\rangle^{(-2)Q}_{l-1,|\phi\rangle^G}$ in the Verma module $V(|\Delta-\mathbf{h},-\mathbf{h}\rangle^G)$. For $l=1$ the upper subdiagram reduces to a chiral couple at level zero (there are no level-zero singular vectors with $|q|=2$).

$$
\begin{array}{ccc}
 & \mathcal{U}_1 \uparrow & \\
|\chi\rangle^{(-1)G}_{l-1,\,|\Delta-\mathbf{h},-\mathbf{h}\rangle^G} & \xrightarrow{\mathcal{Q}_0} & |\chi\rangle^{(-2)Q}_{l-1,\,|\Delta-\mathbf{h},-\mathbf{h}\rangle^G} \\
\mathcal{A}\updownarrow & & \updownarrow\mathcal{A} \\
|\chi\rangle^{(1)Q}_{l-1,\,|\Delta-\mathbf{h},\,\mathbf{h}-\frac{\mathbf{c}}{3}\rangle^Q} & \xrightarrow{\mathcal{G}_0} & |\chi\rangle^{(2)G}_{l-1,\,|\Delta-\mathbf{h},\,\mathbf{h}-\frac{\mathbf{c}}{3}\rangle^Q} \\
 & \mathcal{U}_1 \uparrow & \\
|\chi\rangle^{(1)G}_{l,\,|\Delta,\,\mathbf{h}\rangle^G} & \xrightarrow{\mathcal{Q}_0} & |\chi\rangle^{(0)Q}_{l,\,|\Delta,\,\mathbf{h}\rangle^G} \\
\mathcal{A}\updownarrow & & \updownarrow\mathcal{A} \\
|\chi\rangle^{(-1)Q}_{l,\,|\Delta,-\mathbf{h}-\frac{\mathbf{c}}{3}\rangle^Q} & \xrightarrow{\mathcal{G}_0} & |\chi\rangle^{(0)G}_{l,\,|\Delta,-\mathbf{h}-\frac{\mathbf{c}}{3}\rangle^Q}
\end{array}
\tag{5.2}
$$

We can repeat the same analysis as for the previous skeleton-family, for the particular cases $\Delta=-l$ and $\Delta-\mathbf{h}=1-l$. Now we have to take into account, however, that the cascade effect forbids chiral singular vectors of types $|\chi\rangle^{(1)G,Q}_{|\phi\rangle^G}$, $|\chi\rangle^{(-1)G,Q}_{|\phi\rangle^Q}$, $|\chi\rangle^{(2)G,Q}_{|\phi\rangle^G}$ and $|\chi\rangle^{(-2)G,Q}_{|\phi\rangle^Q}$ which otherwise could appear associated to the skeleton-family (5.2).

The second kind of generic families contains therefore the remaining eight generic types of singular vectors: $|\chi\rangle^{(0)G}_{|\phi\rangle^Q}$, $|\chi\rangle^{(0)Q}_{|\phi\rangle^G}$, $|\chi\rangle^{(1)G}_{|\phi\rangle^G}$ and $|\chi\rangle^{(-1)Q}_{|\phi\rangle^Q}$, at level l, plus $|\chi\rangle^{(1)Q}_{|\phi\rangle^Q}$, $|\chi\rangle^{(-1)G}_{|\phi\rangle^G}$, $|\chi\rangle^{(2)G}_{|\phi\rangle^Q}$ and $|\chi\rangle^{(-2)Q}_{|\phi\rangle^G}$ at level $l-1$, plus the four chiral types of singular vectors as particular cases. The untwisting of the singular vectors of types $|\chi\rangle^{(1)G}_{|\phi\rangle^G}$ and $|\chi\rangle^{(-1)G,Q}_{|\phi\rangle^G}$ produces charged singular vectors of the NS algebra. As a result, all the singular vectors of the second kind of generic families can be mapped to charged NS singular vectors.

Observe that the four types of chiral singular vectors can appear as particular cases, both in the first kind and in the second kind of generic families. They produce therefore the intersection of these two kinds of families, which in the absence of chiral singular vectors are completely disconnected from each other.

Enlargements of the generic families

When the conformal weight of the primary state $|\phi\rangle^G$ or $|\phi\rangle^Q$ is different from zero the highest weight of the Verma module is degenerate, so that one has the choice to express the primaries of type $|\phi\rangle^Q$ as $|\phi\rangle^Q = \mathcal{Q}_0\,|\phi\rangle^G$ (*i.e.* $|\Delta, \mathbf{h}\rangle^Q = \mathcal{Q}_0\,|\Delta, \mathbf{h}+1\rangle^G$), or the other way around using $\mathcal{G}_0$. As a result, for $\Delta \neq 0$ the singular vectors of types $|\chi\rangle^{(q)G}_{|\Delta,\mathbf{h}\rangle^Q}$ and $|\chi\rangle^{(q)Q}_{|\Delta,\mathbf{h}\rangle^G}$ can be expressed as singular vectors of types $|\chi\rangle^{(q-1)G}_{|\Delta,\mathbf{h}+1\rangle^G}$ and $|\chi\rangle^{(q+1)Q}_{|\Delta,\mathbf{h}-1\rangle^Q}$, respectively (observe that $\mathbf{h} \to \mathbf{h} \pm 1$ and $q \to q \mp 1$), and the uncharged chiral singular vectors (for which $\Delta \neq 0$) can be expressed always as charged chiral singular vectors.

This brings about important consequences because the spectral flow mappings $\mathcal{U}_{\pm 1}$ distinguish drastically between a given type of singular vector and the equivalent type. On the one hand, there are arrows $\mathcal{U}_{\pm 1}$ coming in or out of the singular vectors of types $|\chi\rangle^{(q)G}_{|\phi\rangle^G}$ and $|\chi\rangle^{(q)Q}_{|\phi\rangle^Q}$ while there are none for the singular vectors of types $|\chi\rangle^{(q)G}_{|\phi\rangle^Q}$ and $|\chi\rangle^{(q)Q}_{|\phi\rangle^G}$. On the other hand, there are different arrows $\mathcal{U}_{\pm 1}$ coming in or out of the uncharged chiral singular vectors and of their charged partners.

Therefore if the conformal weights of the four Verma modules involved in diagrams (5.1) and (5.2) are different from zero, *i.e.* $\Delta \neq 0$ and $\Delta \neq \mathbf{h}$, then one can express the corresponding singular vectors in the way shown in diagrams (5.3) and (5.4). In this case the first kind of skeleton-families can be viewed as those consisting of the uncharged singular vectors of the types $|\chi\rangle^{(0)G}_{|\phi\rangle^G}$ and $|\chi\rangle^{(0)Q}_{|\phi\rangle^Q}$, including the chiral ones $|\chi\rangle^{(0)G,Q}_{|\phi\rangle^G}$ and $|\chi\rangle^{(0)G,Q}_{|\phi\rangle^Q}$ as particular cases, while the second kind of skeleton-families can be viewed as those consisting of the charged singular vectors of the types $|\chi\rangle^{(1)Q}_{|\phi\rangle^Q}$, $|\chi\rangle^{(1)G}_{|\phi\rangle^G}$, $|\chi\rangle^{(-1)G}_{|\phi\rangle^G}$ and $|\chi\rangle^{(-1)Q}_{|\phi\rangle^Q}$, including the chiral ones $|\chi\rangle^{(1)G,Q}_{|\phi\rangle^Q}$ and $|\chi\rangle^{(-1)G,Q}_{|\phi\rangle^G}$ as particular cases.

Moreover, now one can attach arrows $\mathcal{U}_{\pm 1}$ to each of these singular vectors, and complete subfamilies with them, since they belong to the appropriate types. To be precise, using $\mathcal{U}_{\pm 1}$ one can attach a subfamily of singular vectors, *i.e.* either a four-member "box" or a chiral couple, to every pair of singular vectors of types $|\chi\rangle^{(q)G}_{|\phi\rangle^G}$ and $|\chi\rangle^{(-q)Q}_{|\phi\rangle^Q}$ connected by $\mathcal{A}$. These arrows $\mathcal{U}_{\pm 1}$, that we draw horizontally on the right of diagrams (5.3) and (5.4), do not return to the left of the diagrams, consequently, unlike in the case of the vertical upper-most arrows which return to the bottom of the diagrams.

Observe that the chiral singular vectors produce the intersection of the two kinds of skeleton-families also because the uncharged chiral singular vectors are equivalent to charged chiral singular vectors.

Sequences of attachements

Starting with the subfamily which contain the $\mathcal{G}_0$-closed singular vector $|\chi\rangle^{(q)G}_{l,|\Delta,\mathbf{h}\rangle^G}$ one can distinguish two different sequences of attachements: the sequence starting as $\mathcal{U}_1\ |\Delta,\mathbf{h}\rangle^G \rightarrow |\Delta-\mathbf{h},\mathbf{h}-\mathbf{c}/3\rangle^Q$, which will be denoted as sequence A, and the sequence starting as $\mathcal{U}_1\ |\Delta,-\mathbf{h}-\mathbf{c}/3+1\rangle^G \rightarrow |\Delta+\mathbf{h}+\mathbf{c}/3-1,-\mathbf{h}-2\mathbf{c}/3+1\rangle^Q$, which will be denoted as sequence B.

Let us analyze in some detail sequences A and B, with the help of diagrams (5.1), (5.2), (5.3) and (5.4), focusing on the singular vectors of type $|\chi\rangle^{(q)G}_{|\phi\rangle^G}$. Sequence A starts

$$
\begin{array}{lccccc}
 & \mathcal{U}_1 \uparrow & & & & \\
 & |\chi\rangle^{(0)G}_{l,\,|\Delta-\mathbf{h},-\mathbf{h}\rangle^G} & \xrightarrow{\mathcal{Q}_0} & |\chi\rangle^{(0)Q}_{l,\,|\Delta-\mathbf{h},-\mathbf{h}-1\rangle^Q} & \xrightarrow{\mathcal{U}_{-1}} & \\
\text{for} \Delta \neq \mathbf{h} & \mathcal{A} \updownarrow & & \updownarrow \mathcal{A} & & \\
 & |\chi\rangle^{(0)Q}_{l,\,|\Delta-\mathbf{h},\,\mathbf{h}-\frac{\mathbf{c}}{3}\rangle^Q} & \xrightarrow{\mathcal{G}_0} & |\chi\rangle^{(0)G}_{l,\,|\Delta-\mathbf{h},\,\mathbf{h}-\frac{\mathbf{c}}{3}+1\rangle^G} & \xrightarrow{\mathcal{U}_1} & \\
 & \mathcal{U}_1 \uparrow & & & & \\
 & |\chi\rangle^{(0)G}_{l,\,|\Delta,\,\mathbf{h}\rangle^G} & \xrightarrow{\mathcal{Q}_0} & |\chi\rangle^{(0)Q}_{l,\,|\Delta,\,\mathbf{h}-1\rangle^Q} & \xrightarrow{\mathcal{U}_{-1}} & \\
\text{for} \Delta \neq 0 & \mathcal{A} \updownarrow & & \updownarrow \mathcal{A} & & \\
 & |\chi\rangle^{(0)Q}_{l,\,|\Delta,-\mathbf{h}-\frac{\mathbf{c}}{3}\rangle^Q} & \xrightarrow{\mathcal{G}_0} & |\chi\rangle^{(0)G}_{l,\,|\Delta,-\mathbf{h}-\frac{\mathbf{c}}{3}+1\rangle^G} & \xrightarrow{\mathcal{U}_1} &
\end{array}
\qquad (5.3)
$$

with the mapping $\mathcal{U}_1\,|\chi\rangle^{(q)G}_{l,\,|\Delta,\,\mathbf{h}\rangle^G} \rightarrow |\chi\rangle^{(q)Q}_{l-q,\,|\Delta-\mathbf{h},\,\mathbf{h}-\mathbf{c}/3\rangle^Q}$ (a chiral singular vector $|\chi\rangle^{(q)G,Q}_{l,\,|\Delta,\,\mathbf{h}\rangle^G}$ can also be taken instead). Then using $\mathcal{A}$, $\mathcal{G}_0$ and $\mathcal{Q}_0$ one obtains a second subfamily attached to the former by $\mathcal{U}_1$ (more precisely $\mathcal{U}_{\pm1}$ since we can invert the arrow). Next, one looks for further singular vectors of type $|\chi\rangle^{(q)G}_{|\phi\rangle^G}$, in the second subfamily, to which attach new arrows $\mathcal{U}_1$. The upper-most singular vector of this type does not lead to a new subfamily, but to the previous subfamily, because $\mathcal{U}_1\mathcal{A}\mathcal{U}_1\mathcal{A} = \mathbf{I}$, as pointed out before. For this reason, if the second subfamily consists of a chiral couple this sequence is finished. If the second subfamily is a four-member "box", like in the diagrams above, there are three different possibilities depending on whether the conformal weights $\Delta-\mathbf{h}$ and $\Delta-\mathbf{h}+l-q$, of the primary states and the singular vectors, respectively, are zero or different from zero. If $\Delta-\mathbf{h}\neq 0$ and $\Delta-\mathbf{h}\neq q-l$, then the singular vector of type

$$
\begin{array}{lccccc}
 & \mathcal{U}_1 \uparrow & & & & \\
 & |\chi\rangle^{(-1)G}_{l-1,\,|\Delta-\mathbf{h},-\mathbf{h}\rangle^G} & \xrightarrow{\mathcal{Q}_0} & |\chi\rangle^{(-1)Q}_{l-1,\,|\Delta-\mathbf{h},-\mathbf{h}-1\rangle^Q} & \xrightarrow{\mathcal{U}_{-1}} & \\
\text{for}\Delta \neq \mathbf{h} & \mathcal{A} \updownarrow & & \updownarrow \mathcal{A} & & \\
 & |\chi\rangle^{(1)Q}_{l-1,\,|\Delta-\mathbf{h},\,\mathbf{h}-\frac{\mathbf{c}}{3}\rangle^Q} & \xrightarrow{\mathcal{G}_0} & |\chi\rangle^{(1)G}_{l-1,\,|\Delta-\mathbf{h},\,\mathbf{h}-\frac{\mathbf{c}}{3}+1\rangle^G} & \xrightarrow{\mathcal{U}_1} & \\
 & \mathcal{U}_1 \uparrow & & & & \\
 & |\chi\rangle^{(1)G}_{l,\,|\Delta,\,\mathbf{h}\rangle^G} & \xrightarrow{\mathcal{Q}_0} & |\chi\rangle^{(1)Q}_{l,\,|\Delta,\,\mathbf{h}-1\rangle^Q} & \xrightarrow{\mathcal{U}_{-1}} & \\
\text{for}\Delta \neq 0 & \mathcal{A} \updownarrow & & \updownarrow \mathcal{A} & & \\
 & |\chi\rangle^{(-1)Q}_{l,\,|\Delta,-\mathbf{h}-\frac{\mathbf{c}}{3}\rangle^Q} & \xrightarrow{\mathcal{G}_0} & |\chi\rangle^{(-1)G}_{l,\,|\Delta,-\mathbf{h}-\frac{\mathbf{c}}{3}+1\rangle^G} & \xrightarrow{\mathcal{U}_1} &
\end{array}
\qquad (5.4)
$$

$|\chi\rangle^{(q+1)G}_{|\phi\rangle^Q}$ can be expressed as a singular vector of type $|\chi\rangle^{(q)G}_{|\phi\rangle^G}$, as in diagrams (5.3) and (5.4). Namely, $|\chi\rangle^{(q+1)G}_{l-q,\,|\Delta-\mathbf{h},\,\mathbf{h}-\mathbf{c}/3\rangle^Q} = |\chi\rangle^{(q)G}_{l-q,\,|\Delta-\mathbf{h},\,\mathbf{h}-\mathbf{c}/3+1\rangle^G}$. Attaching an arrow $\mathcal{U}_1$ to this singular vector one obtains a new subfamily different from the previous one, although involving the same types of singular vectors which are already in the skeleton-family. If $\Delta - \mathbf{h} = 0$ and $\Delta - \mathbf{h} \neq q - l$, then the singular vector of type $|\chi\rangle^{(q+1)G}_{|\phi\rangle^Q}$ cannot be expressed as a singular vector of type $|\chi\rangle^{(q)G}_{|\phi\rangle^G}$, and the sequence is finished since one cannot attach any more arrows $\mathcal{U}_{\pm1}$. Finally, if $\Delta - \mathbf{h} = q - l$ (and the singular vectors on the left-hand side are not chiral, as we assume) then $\mathcal{Q}_0$ and $\mathcal{G}_0$ produce secondary chiral singular vectors on the right-hand side, to which one can attach new arrows $\mathcal{U}_{\pm1}$.

Repeating the same reasoning n times one finds that sequence A continues or stops after n steps depending on whether the conformal weights $\hat{\Delta}$ and/or $\hat{\Delta} + l - nq$ are zero or different from zero, where

$$
\hat{\Delta} = \Delta - n\mathbf{h} - n(n-1)(\frac{3-\mathbf{c}}{6}) \ , \qquad n \in \mathbf{Z}^+ \qquad (5.5)
$$

Namely, if $\hat{\Delta} \neq 0$ and $\hat{\Delta} \neq nq - l$, the sequence of singular vectors goes on attaching an arrow $\mathcal{U}_1$ to the $\mathcal{G}_0$-closed singular vector $|\chi\rangle^{(q)G}_{l-nq,\,|\hat{\Delta},\,\hat{\mathbf{h}}\rangle^G}$ at level $l - nq$, with $\hat{\mathbf{h}} = \mathbf{h} + n(\frac{3-\mathbf{c}}{3})$. If $\hat{\Delta} = 0$ and $\hat{\Delta} \neq nq - l$ the sequence is finished, whereas if $\hat{\Delta} = nq - l$ the sequence may finish depending on whether or not the $\mathcal{G}_0$-closed singular vector "becomes" chiral, *i.e.* of type $|\chi\rangle^{(q)G,Q}_{l-nq,\,|\hat{\Delta},\,\hat{\mathbf{h}}\rangle^G}$. Starting with charge $q = 1$ singular vectors $|\chi\rangle^{(1)G}_{l,|\Delta,\mathbf{h}\rangle^G}$ this sequence can finish before one of these conditions is met, since $l - nq$ must be positive. Starting with charge $q = -1$ or with uncharged singular vectors, however, this sequence of mappings and attachements of four-member subfamilies only ends if $\hat{\Delta} = 0$

and $\hat{\Delta} \neq nq-l$ or if $\hat{\Delta} = nq-l$ and the corresponding $\mathcal{G}_0$-closed singular vector turns out to be chiral. In addition this sequence has no loops, although individually a given Verma module can appear more than once, in different subfamilies of the sequence (see Appendix C). *Therefore, depending on the values of* Δ, $\mathbf{h}$, *and* $\mathbf{c}$, *sequence A can give rise to an infinitely large family of singular vectors.*

Sequence B starts with the mapping $\mathcal{G}_0\,\mathcal{A}\,|\chi\rangle^{(q)G}_{l,|\Delta,\mathbf{h}\rangle^G} \rightarrow |\chi\rangle^{(-q)G}_{l,|\Delta,-\mathbf{h}-\mathbf{c}/3+1\rangle^G}$, with $\Delta \neq 0$, followed by the mapping $\mathcal{U}_1\,|\chi\rangle^{(-q)G}_{l,|\Delta,-\mathbf{h}-\mathbf{c}/3+1\rangle^G} \rightarrow |\chi\rangle^{(-q)Q}_{l+q,|\Delta+\mathbf{h}+\mathbf{c}/3-1,-\mathbf{h}-2\mathbf{c}/3+1\rangle^Q}$ (diagrams (5.3) and (5.4)). Following similar reasoning as for sequence A one finds that sequence B continues after n steps if the condition $\hat{\Delta} \neq 0$, $\hat{\Delta} \neq -(nq+l)$, with

$$\hat{\Delta} = \Delta + n\mathbf{h} - n(n+1)(\frac{3-\mathbf{c}}{6}) \ , \qquad n \in \mathbf{Z}^+ + \{0\} \tag{5.6}$$

is satisfied. Then one can attach an arrow $\mathcal{U}_1$ to the singular vector $|\chi\rangle^{(-q)G}_{l+nq,|\hat{\Delta},\hat{\mathbf{h}}\rangle^G}$ at level $l+nq$, with $\hat{\mathbf{h}} = -\mathbf{h} + (n+1)(\frac{3-\mathbf{c}}{3})$. Sequence B stops if $\hat{\Delta} = 0$, $\hat{\Delta} \neq -(nq+l)$, whereas it may stop, although not necessarily, if $\hat{\Delta} = -(nq+l)$. Starting with charge $q=-1$ singular vectors $|\chi\rangle^{(-1)G}_{l,|\Delta,\mathbf{h}\rangle^G}$ this sequence can end before one of these conditions is satisfied, because $l+nq$ must be positive. Starting with charge $q=1$ or with uncharged singular vectors this sequence only ends if $\hat{\Delta} = 0$, $\hat{\Delta} \neq -(nq+l)$, or if $\hat{\Delta} = -(nq+l)$ and the corresponding $\mathcal{G}_0$-closed singular vector turns out to be chiral, *i.e.* of type $|\chi\rangle^{(-q)G,Q}_{l+nq,|\hat{\Delta},\hat{\mathbf{h}}\rangle^G}$. *Therefore, depending on the values of* Δ, $\mathbf{h}$, *and* $\mathbf{c}$, *sequence B can also give rise to an infinitely large family of singular vectors.*

However, unlike in the previous case, this sequence can come back to a given Verma module, *i.e.* it has loops (although these loops do not involve necessarily the same singular vectors, as we will see). Namely, for $\mathbf{h} = (n+1)(\frac{3-\mathbf{c}}{6})$ one finds $\hat{\Delta} = \Delta$ and $\hat{\mathbf{h}} = \mathbf{h}$ after n steps.

Notice that sequences A and B are complementary in that sequence A has the bound $l - nq \geq 0$, which only applies to charge $q=1$ singular vectors, whereas sequence B has the bound $l+nq \geq 0$, which only applies to charge $q=-1$ singular vectors.

Families of no-label singular vectors in generic Verma modules

The no-label types of singular vectors in generic Verma modules are decoupled from the other types in the sense that $\mathcal{A}$ transforms the no-label singular vectors into each other, and $\mathcal{U}_{\pm 1}$ transforms these singular vectors into states that are not singular. The action of $\mathcal{G}_0$ and $\mathcal{Q}_0$, however, produces $\mathcal{G}_0$-closed and $\mathcal{Q}_0$-closed secondary singular vectors, respectively. Therefore we can distinguish two kinds of skeleton-families involving no-label singular vectors: those containing $|\chi\rangle^{(0)}_{|\phi\rangle^G}$ and $|\chi\rangle^{(0)}_{|\phi\rangle^Q}$ and those containing $|\chi\rangle^{(-1)}_{|\phi\rangle^G}$ and $|\chi\rangle^{(1)}_{|\phi\rangle^Q}$. However the singular vectors of types $|\chi\rangle^{(1)}_{|\phi\rangle^Q}$ and $|\chi\rangle^{(-1)}_{|\phi\rangle^G}$ are always equivalent to singular vectors of types $|\chi\rangle^{(0)}_{|\phi\rangle^G}$ and $|\chi\rangle^{(0)}_{|\phi\rangle^Q}$, respectively, so that there is only one skeleton-family left.

Families of singular vectors in no-label Verma modules

The singular vectors in no-label Verma modules $V(|0,\mathbf{h}\rangle)$ are organized into four-member families, like in the case of chiral Verma modules. However, in no-label Verma modules there are two different kinds of four-member families instead of one, as shown in diagram (5.7). These two kinds of families exist already at level 1 (see Appendix B). In addition, at level zero the families containing the uncharged singular vectors reduce to two uncharged chiral singular vectors whereas the other kind of families does not exist.

$$
\begin{array}{ccc}
|\chi\rangle^{(0)G}_{l,\,|0,\,\mathbf{h}\rangle} & \xrightarrow{\mathcal{Q}_0} & |\chi\rangle^{(-1)Q}_{l,\,|0,\,\mathbf{h}\rangle} \\
\mathcal{A}\updownarrow & & \updownarrow\mathcal{A} \\
|\chi\rangle^{(0)Q}_{l,\,|0,-\mathbf{h}-\frac{\mathbf{c}}{3}\rangle} & \xrightarrow{\mathcal{G}_0} & |\chi\rangle^{(1)G}_{l,\,|0,-\mathbf{h}-\frac{\mathbf{c}}{3}\rangle}
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
|\chi\rangle^{(2)G}_{l,\,|0,\,\mathbf{h}\rangle} & \xrightarrow{\mathcal{Q}_0} & |\chi\rangle^{(1)Q}_{l,\,|0,\,\mathbf{h}\rangle} \\
\mathcal{A}\updownarrow & & \updownarrow\mathcal{A} \\
|\chi\rangle^{(-2)Q}_{l,\,|0,-\mathbf{h}-\frac{\mathbf{c}}{3}\rangle} & \xrightarrow{\mathcal{G}_0} & |\chi\rangle^{(-1)G}_{l,\,|0,-\mathbf{h}-\frac{\mathbf{c}}{3}\rangle}
\end{array}
\tag{5.7}
$$

5.2 Spectrum of Δ and $\mathbf{h}$

Now let us consider the spectra of conformal weights Δ and U(1) charges $\mathbf{h}$ corresponding to the complete Verma modules which contain singular vectors.

Generic singular vectors

For the case of generic singular vectors the spectra of Δ and $\mathbf{h}$ can be obtained easily from the spectra corresponding to the NS algebra, given by the zeroes of the determinant formula [25], [26], [27]. The argument goes as follows. From the determinant formula one deduces that the Verma module of the NS algebra $V_{NS}(\Delta',\mathbf{h}')$ has (at least) one uncharged singular vector, at level $l' = \frac{rs}{2}$, if $(\Delta',\mathbf{h}')$ lies on the quadratic vanishing surface $f_{r,s}(\Delta',\mathbf{h}') = 0$, where

$$
f_{r,s}(\Delta',\mathbf{h}') = -2t\Delta' - \mathbf{h}'^2 - \frac{1}{4}t^2 + \frac{1}{4}(s-tr)^2 \quad , \quad r \in \mathbf{Z}^+,\ s \in 2\mathbf{Z}^+, \tag{5.8}
$$

with $t = (3-\mathbf{c})/3$. If $(\Delta',\mathbf{h}')$ lies on the vanishing plane $g_k(\Delta',\mathbf{h}') = 0$ instead, where

$$
g_k(\Delta',\mathbf{h}') = 2\Delta' - 2k\mathbf{h}' - t(k^2 - \frac{1}{4}) \quad , \quad k \in \mathbf{Z} + 1/2, \tag{5.9}
$$

then $V_{NS}(\Delta',\mathbf{h}')$ has (at least) one charged singular vector, at level $l' = |k|$, with relative charge $q' = sgn(k)$. On the other hand, by inspecting the twists T_{W1} (2.2) and T_{W2} (2.3), one finds that they transform the primary states and singular vectors of the NS algebra,

with (L_0, H_0) eigenvalues $(\Delta', \mathbf{h}')$, into $\mathcal{G}_0$-closed primary states and $\mathcal{G}_0$-closed singular vectors of the Topological algebra with $(\mathcal{L}_0, \mathcal{H}_0)$ eigenvalues $(\Delta, \mathbf{h})$, where $\Delta = \Delta' \pm \frac{\mathbf{h}'}{2}$ and $\mathbf{h} = \pm\mathbf{h}'$. Consequently, the level and the relative U(1) charge of the topological singular vectors are given by $l = l' \pm \frac{q'}{2}$ and $q = \pm q'$ respectively.

It is convenient to rewrite $f_{r,s}(\Delta', \mathbf{h}')$ and $g_k(\Delta', \mathbf{h}')$ in terms of the topological $(\mathcal{L}_0, \mathcal{H}_0)$ eigenvalues, using $(\Delta', \mathbf{h}') = (\Delta - \frac{\mathbf{h}}{2}, \pm\mathbf{h})$, resulting in

$$f_{r,s}(\Delta, \mathbf{h}) = -2t\Delta + t\mathbf{h} - \mathbf{h}^2 - \frac{1}{4}t^2 + \frac{1}{4}(s - tr)^2 \quad , \quad r \in \mathbf{Z}^+,\ s \in 2\mathbf{Z}^+ \tag{5.10}$$

and

$$g_k(\Delta, \mathbf{h}) = 2\Delta - \mathbf{h} \mp 2k\mathbf{h} - t(k^2 - \frac{1}{4}) \quad , \quad k \in \mathbf{Z} + 1/2 \tag{5.11}$$

As a consequence if $f_{r,s}(\Delta, \mathbf{h}) = 0$ the topological Verma module $V(|\Delta, \mathbf{h}\rangle^G)$ has (at least) one uncharged $\mathcal{G}_0$-closed singular vector $|\chi\rangle^{(0)G}_{|\phi\rangle^G}$, at level $l = \frac{rs}{2}$. If $g_k(\Delta, \mathbf{h}) = 0$ instead, then the topological Verma module $V(|\Delta, \mathbf{h}\rangle^G)$ has (at least) one charged $\mathcal{G}_0$-closed singular vector $|\chi\rangle^{(q)G}_{|\phi\rangle^G}$, at level $l = |k| \pm \frac{q'}{2}$ with relative charge $q = \pm q'$, where $q' = sgn(k)$. In addition, the singular vectors of type $|\chi\rangle^{(-1)Q}_{|\phi\rangle^G}$ are in the same Verma modules and at the same levels as the singular vectors of type $|\chi\rangle^{(0)G}_{|\phi\rangle^G}$, and the same happens with the singular vectors of types $|\chi\rangle^{(0)Q}_{|\phi\rangle^G}$ and $|\chi\rangle^{(1)G}_{|\phi\rangle^G}$, on the one side, and with the singular vectors of types $|\chi\rangle^{(-2)Q}_{|\phi\rangle^G}$ and $|\chi\rangle^{(-1)G}_{|\phi\rangle^G}$, on the other.

As to the singular vectors built on $\mathcal{Q}_0$-closed topological primaries, one gets the corresponding spectra straightforwardly just by applying the spectral flow automorphism $\mathcal{A}$ (3.3) which interchanges $G \leftrightarrow Q$, $q \leftrightarrow -q$ and $\mathbf{h} \leftrightarrow -\mathbf{h} - \frac{\mathbf{c}}{3}$. For example, the singular vectors of types $|\chi\rangle^{(0)Q}_{|\phi\rangle^Q}$ and $|\chi\rangle^{(1)G}_{|\phi\rangle^Q}$ are at the same level in the Verma module $V(|\Delta, \hat{\mathbf{h}}\rangle^Q)$ satisfying $f_{r,s}(\Delta, -\hat{\mathbf{h}} - \frac{\mathbf{c}}{3}) = 0$ in (5.10), and similarly the remaining cases.

Chiral singular vectors

Chiral singular vectors can be viewed as particular cases of $\mathcal{G}_0$-closed singular vectors which become chiral when $\Delta + l = 0$, but they can also be viewed as particular cases of $\mathcal{Q}_0$-closed singular vectors which become chiral when $\Delta + l = 0$. As a consequence, the spectra of Δ and $\mathbf{h}$ corresponding to chiral singular vectors of type $|\chi\rangle^{(0)G,Q}_{l,|\Delta,\mathbf{h}\rangle^G}$ must lie on the intersection of the spectra corresponding to singular vectors of types $|\chi\rangle^{(0)G}_{l,|\Delta,\mathbf{h}\rangle^G}$ and $|\chi\rangle^{(0)Q}_{l,|\Delta,\mathbf{h}\rangle^G}$, with $\Delta = -l$. These are given by the solutions to the quadratic vanishing surface $f_{r,s}(-l, \mathbf{h}) = 0$, and the solutions to the vanishing plane $g_{l-1/2}(-l, \mathbf{h}) = 0$, respectively (the type $|\chi\rangle^{(0)Q}_{l,|\phi\rangle^G}$ has the same spectra as the type $|\chi\rangle^{(1)G}_{l,|\phi\rangle^G}$). The intersection of $f_{r,s}(-l, \mathbf{h}) = 0$ (5.10) and $g_{l-1/2}(-l, \mathbf{h}) = 0$ (5.11) gives

$$\mathbf{h} = \frac{t}{2}(1-l) - 1, \qquad (s-tr)^2 = (2-tl)^2 \tag{5.12}$$

with $l = \frac{rs}{2} > 0$. The obvious general solution for any t is $s = 2,\ r = l$, and there is also the particular solution $r = 2/t,\ s = lt$, which only makes sense for the values of t such that $r \in \mathbf{Z}^+$ and $s \in 2\mathbf{Z}^+$. At level zero there are no uncharged chiral singular vectors (they exist only on no-label primaries).

Similarly, the spectra corresponding to chiral singular vectors of type $|\chi\rangle^{(-1)G,Q}_{l,\,|\Delta,\,\mathbf{h}\rangle^G}$ must lie on the intersection of the spectra corresponding to singular vectors of types $|\chi\rangle^{(-1)G}_{l,\,|\Delta,\,\mathbf{h}\rangle^G}$ and $|\chi\rangle^{(-1)Q}_{l,\,|\Delta,\,\mathbf{h}\rangle^G}$, with $\Delta = -l$, given by the solutions to $g_{-(l+1/2)}(-l,\mathbf{h}) = 0$, and the solutions to $f_{r,s}(-l,\mathbf{h}) = 0$, respectively (the type $|\chi\rangle^{(-1)Q}_{l,\,|\phi\rangle^G}$ has the same spectra as the type $|\chi\rangle^{(0)G}_{l,\,|\phi\rangle^G}$). The intersection of $f_{r,s}(-l,\mathbf{h}) = 0$ and $g_{-(l+1/2)}(-l,\mathbf{h}) = 0$ gives

$$\mathbf{h} = \frac{t}{2}(1+l) + 1, \qquad (s-tr)^2 = (2-tl)^2 \tag{5.13}$$

with $l = \frac{rs}{2} > 0$. As before, the general solution is $s = 2,\ r = l$, for any t, and there is also the particular solution $r = 2/t,\ s = lt$. For $l = 0$ there are charged chiral singular vectors $|\chi\rangle^{(-1)G,Q}_{0,\,|0,\,\mathbf{h}\rangle^G} = \mathcal{Q}_0|0,\,\mathbf{h}\rangle^G$ for every value of $\mathbf{h}$.

The untwisting of equations (5.12) and (5.13) give the same equations back, with T_{W2} reversing the sign of $\mathbf{h}$. Therefore $\mathbf{h}$ and $(-\mathbf{h})$ in (5.12) give the spectrum corresponding to *uncharged antichiral and chiral singular vectors of the NS algebra* at level l, respectively, while $\mathbf{h}$ and $(-\mathbf{h})$ in (5.13) give the spectrum corresponding to *charge* $q = -1$ *antichiral and charge* $q = 1$ *chiral singular vectors of the NS algebra* at level $l + \frac{1}{2}$, respectively. As far as we know these results for the chiral and antichiral singular vectors of the NS algebra were unknown.

Other types of singular vectors

Regarding singular vectors in no-label Verma modules and no-label singular vectors in generic Verma modules, we ignore the corresponding spectra of Δ and/or $\mathbf{h}$. The existing singular vectors at level 1, shown in Appendix B, suggest however that the spectrum of $\mathbf{h}$ for no-label Verma modules $V(|0,\mathbf{h}\rangle)$ is the "sum" of the spectra corresponding to the Verma modules $V(|0,\mathbf{h}\rangle^G)$ and $V(|0,\mathbf{h}\rangle^Q)$, and also suggest that the no-label singular vectors are very scarce (at level 1 they only exist for $\mathbf{c} = -3$).

6 Internal Mappings and Dörrzapf Pairs

An interesting issue is to analyze under which conditions the chains of mappings involving $\mathcal{A}$ and $\mathcal{U}_{\pm 1}$ act *inside* a Verma module. By inspecting diagrams (5.1) and (5.2) one sees only two possibilities: the Verma modules $V(|\Delta, \mathbf{h}\rangle^G)$ and $V(|\Delta - \mathbf{h}, -\mathbf{h}\rangle^G)$, connected by $\mathcal{A}\mathcal{U}_1$, coincide for $\mathbf{h} = 0$, and so do the Verma modules $V(|\Delta, -\mathbf{h} - \mathbf{c}/3\rangle^Q)$ and $V(|\Delta - \mathbf{h}, \mathbf{h} - \mathbf{c}/3\rangle^Q)$, related by $\mathcal{U}_1\mathcal{A}$. In diagrams (5.3) and (5.4) there are more possibilities. For example, the Verma modules $V(|\Delta, \mathbf{h}\rangle^G)$ and $V(|\Delta, -\mathbf{h} - \mathbf{c}/3 + 1\rangle^G)$ coincide for $\mathbf{h} = \frac{3-\mathbf{c}}{6}$ while the Verma modules $V(|\Delta - \mathbf{h}, -\mathbf{h}\rangle^G)$ and $V(|\Delta - \mathbf{h}, \mathbf{h} - \mathbf{c}/3 + 1\rangle^G)$ coincide for $\mathbf{h} = \frac{\mathbf{c}-3}{6}$.

The case $\mathbf{h} = \frac{3-\mathbf{c}}{6} = \frac{t}{2}$ is the simplest example of a loop associated to sequence B, corresponding to $n = 0$. In general, for $\mathbf{h} = (n+1)t/2 \;\; (n = 0, 1, 2....)$ one finds $\hat{\Delta} = \Delta$ and $\hat{\mathbf{h}} = \mathbf{h}$ after n attachements of four-member subfamilies. For this value of $\mathbf{h}$ and $\Delta_{r,s} = \frac{1}{8t}[(s - tr)^2 - n^2t^2]$, satisfying $f_{r,s}(\Delta, \mathbf{h}) = 0$ in eq. (5.10), the loop of mappings connect in the same Verma module and at the same level two singular vectors of type $|\chi\rangle^{(0)G}_{l,|\phi\rangle^G}$ and two singular vectors of type $|\chi\rangle^{(-1)Q}_{l,|\phi\rangle^G}$. For $\mathbf{h} = (n+1)t/2$ eqns. (5.12) and (5.13) give $n+l = -\frac{2}{t}$ and $n-l = \frac{2}{t}$ respectively. Therefore for these values the mappings connect a couple of chiral singular vectors $|\chi\rangle^{(0)G,Q}_{l,|\phi\rangle^G}$ and $|\chi\rangle^{(-1)G,Q}_{l,|\phi\rangle^G}$ instead. Finally, for $\mathbf{h} = (n+1)t/2$ and $\Delta_k = \frac{nt}{4}(1 \pm 2k) + \frac{t}{2}(k \pm \frac{1}{2})^2$, satisfying $g_k(\Delta, \mathbf{h}) = 0$ in eq. (5.11), the mappings connect four singular vectors of different types in the same Verma module: $|\chi\rangle^{(1)G}_{l,|\phi\rangle^G}$ and $|\chi\rangle^{(0)Q}_{l,|\phi\rangle^G}$ at level l, and $|\chi\rangle^{(-1)G}_{l+n,|\phi\rangle^G}$ and $|\chi\rangle^{(-2)Q}_{l+n,|\phi\rangle^G}$ at level $l+n$.

The question naturally arises whether the two uncharged singular vectors $|\chi\rangle^{(0)G}_{l,|\phi\rangle^G}$ and the two charged singular vectors $|\chi\rangle^{(-1)Q}_{l,|\phi\rangle^G}$, which are together in the same Verma module at the same level, coincide (up to constants) or they are linearly independent. In the NS algebra, the possibility that two uncharged singular vectors, at the same level, in the same Verma module, may be linearly independent, was discovered by Dörrzapf in ref. [9]. Under the topological twistings T_{W1} (2.2) and T_{W2} (2.3) this possibility extends also to the topological singular vectors of type $|\chi\rangle^{(0)G}_{|\phi\rangle^G}$, and to all the types related *necessarily* to them via the mappings that we have analyzed; *i.e.* the three types of generic singular vectors involved, together with $|\chi\rangle^{(0)G}_{|\phi\rangle^G}$, in the first kind of generic families, depicted in diagram (5.1): $|\chi\rangle^{(-1)Q}_{|\phi\rangle^G}$, $|\chi\rangle^{(0)Q}_{|\phi\rangle^Q}$ and $|\chi\rangle^{(1)G}_{|\phi\rangle^Q}$. This possibility does not extend, however, to the eight types of generic singular vectors involved in the second kind of generic families, depicted in diagram (5.2) because they are related *necessarily* to the charged singular vectors of the NS algebra, and Dörrzapf proved that for these singular vectors there are no such two dimensional spaces. As to the chiral types of singular vectors, they do not admit two dimensional spaces [22], in spite of the fact that they may appear in generic families of the first kind as particular cases. However, chiral singular vectors may appear as partners of $\mathcal{G}_0$-closed or $\mathcal{Q}_0$-closed singular vectors in Dörrzapf pairs since they are just particular cases of $\mathcal{G}_0$-closed and $\mathcal{Q}_0$-closed singular vectors (see Appendix C).

Observe that the loops of sequence B, mapping singular vectors into singular vectors inside the same Verma module, connect singular vectors of the same type, at the same

level, only for the types which admit two dimensional spaces. This fact strongly suggests that these loops of mappings may transform, under appropriate conditions, a given singular vector of any of the types $|\chi\rangle^{(0)G}_{|\phi\rangle^G}$, $|\chi\rangle^{(-1)Q}_{|\phi\rangle^G}$, $|\chi\rangle^{(0)Q}_{|\phi\rangle^Q}$, or $|\chi\rangle^{(1)G}_{|\phi\rangle^Q}$, into a linearly independent singular vector of exactly the same type, at the same level.

The Dörrzapf conditions for the appearance of two linearly independent uncharged NS singular vectors at the same level, in the same Verma module, consist of the simultaneous vanishing of two curves, $\epsilon^+_{r,s}(t,\mathbf{h})=0$ and $\epsilon^-_{r,s}(t,\mathbf{h})=0$, given by

$$\epsilon^{\pm}_{r,s}(t,\mathbf{h})=\prod_{m=1}^{r}(\pm\frac{s-rt}{2t}+\frac{\mathbf{h}}{t}\mp\frac{1}{2}\pm m)\ \ ,\ r\in\mathbf{Z}^+,\ \ s\in 2\mathbf{Z}^+ \qquad (6.1)$$

where $t=\frac{3-\mathbf{c}}{3}$ and the level of the singular vector is $l=rs/2$. The topological twists T_{W1} (2.2) and T_{W2} (2.3) let these conditions invariant, extending the existence of the two dimensional space of singular vectors to the topological singular vectors of types $|\chi\rangle^{(0)G}_{|\phi\rangle^G}$, $|\chi\rangle^{(-1)Q}_{|\phi\rangle^G}$, $|\chi\rangle^{(1)G}_{|\phi\rangle^Q}$ and $|\chi\rangle^{(0)Q}_{|\phi\rangle^Q}$, as we pointed out.

Let us analyze the conditions $\epsilon^+_{r,s}(t,\mathbf{h})=0$ and $\epsilon^-_{r,s}(t,\mathbf{h})=0$ for $\mathbf{h}=(n+1)t/2$, which correspond to the loops in sequence B. One gets the expressions:

$$\epsilon^+_{r,s}(t,(n+1)t/2)=\prod_{m=1}^{r}(\frac{s-rt}{2t}+\frac{n}{2}+m) \qquad (6.2)$$

$$\epsilon^-_{r,s}(t,(n+1)t/2)=\prod_{m=1}^{r}(\frac{rt-s}{2t}+\frac{n}{2}+1-m) \qquad (6.3)$$

Inspecting these, one realizes that the vanishing of the term corresponding to $m=m_1$ in the first expression results in the vanishing of the term corresponding to $m=m_2$ in the second expression provided $m_2=m_1+n+1$. This implies that $r\geq n+2$ for the singular vector, but the equality $r=n+2$, $m_1=1$ has no solution for the vanishing of the first term in $\epsilon^+_{r,s}$. Therefore $r>n+2$, implying $l>n+2$ since $l\geq r$.

We see that one has to look at level 3, at least, to check if the loops in sequence B connect “Dörrzapf pairs” of singular vectors, since $l>2$ already for $n=0$ ($\mathbf{h}=t/2$). We have checked that this is indeed the case for the singular vectors which correspond to the solutions to $\epsilon^+_{r,s}(t,t/2)=\epsilon^-_{r,s}(t,t/2)=0$ at level 3 ($r=3$, $s=2$), which are $t=2$ ($\mathbf{c}=-3$) and $t=-2$ ($c=9$). In Appendix C we have written down the pairs of singular vectors of types $|\chi\rangle^{(0)G}_{|\phi\rangle^G}$ and $|\chi\rangle^{(0)Q}_{|\phi\rangle^Q}$ (equivalent to pairs of singular vectors of types $|\chi\rangle^{(1)G}_{|\phi\rangle^Q}$ and $|\chi\rangle^{(-1)Q}_{|\phi\rangle^G}$, respectively), for the case $\mathbf{h}=t/2=-1$, $\mathbf{c}=9$, together with the complete family to which they belong. It turns out that this family contains two more Dörrzapf pairs, which are not connected by any loops of sequence B, and with the particularity that one member in each pair is chiral.

When the singular vectors of types $|\chi\rangle^{(0)G}_{|\phi\rangle^G}$, $|\chi\rangle^{(-1)Q}_{|\phi\rangle^G}$, $|\chi\rangle^{(0)Q}_{|\phi\rangle^Q}$ and $|\chi\rangle^{(1)G}_{|\phi\rangle^Q}$ do not satisfy the Dörrzapf conditions (6.1), then they must be mapped back to themselves (up to constants) by the loops of sequence B, and by any other mappings acting inside a given Verma module. In Appendix B we show an example of a singular vector of type $|\chi\rangle^{(0)G}_{|\phi\rangle^G}$ at level 1, with $\mathbf{h} = t/2$, which comes back to itself under the first loop of sequence B.

To finish, we conjecture that the two partners in every Dörrzapf pair of singular vectors of types $|\chi\rangle^{(0)G}_{l,|\phi\rangle^G}$, $|\chi\rangle^{(-1)Q}_{l,|\phi\rangle^G}$, $|\chi\rangle^{(1)G}_{|\phi\rangle^Q}$ and $|\chi\rangle^{(0)Q}_{|\phi\rangle^Q}$, are connected by the mappings made out of $\mathcal{A}$, $\mathcal{U}_{\pm 1}$, $\mathcal{G}_0$ and $\mathcal{Q}_0$, and therefore they belong to the same family.

7 Conclusions and Final Remarks

We have analyzed several issues concerning the singular vectors of the Topological algebra, considering chiral Verma modules as well as complete Verma modules.

First, we have investigated which are the different types of topological singular vectors which may exist, taking into account the relative U(1) charge q and the BRST-invariance properties of the singular vectors themselves and of the primary states on which they are built (this is of utmost importance). We have identified an algebraic mechanism, "the cascade effect", which provides a necessary, although not sufficient, condition for the existence of a given type of singular vector. In chiral Verma modules $V(|0, \mathbf{h}\rangle^{G,Q})$, *i.e.* built on chiral primaries, there are only four types of singular vectors allowed by the cascade effect, all four with $|q| \leq 1$, as shown in table (2.5). By explicit construction we know that the four types exist at low levels, even at level 1. In complete Verma modules, built on topological primaries without additional constraints other than the ones imposed by the algebra, one finds a total of twenty-nine different types of singular vectors allowed by the cascade effect, all of them with $|q| \leq 2$, as shown in tables (2.6), (2.7) and (2.8). Twenty of these types (twelve generic, four chiral and four no-label) correspond to singular vectors in generic Verma modules, $V(|\Delta, \mathbf{h}\rangle^G)$ and $V(|\Delta, \mathbf{h}\rangle^Q)$, and the remaining nine types correspond to singular vectors in no-label Verma modules $V(|0, \mathbf{h}\rangle)$. Twenty-eight of these types can be constructed already at level 1 and one type only exists at level zero. In addition, the twelve generic types and the four chiral types of singular vectors in generic Verma modules can be mapped to the singular vectors of the NS algebra and therefore they must necessarily exist.

Second, we have analyzed a set of mappings between the topological singular vectors: the spectral flow mappings $\mathcal{A}$ and $\mathcal{U}_{\pm 1}$, and the action of the zero modes $\mathcal{G}_0$ and $\mathcal{Q}_0$. While $\mathcal{G}_0$ and $\mathcal{Q}_0$ act inside a given Verma module, $\mathcal{A}$ and $\mathcal{U}_{\pm 1}$ interpolate between different Verma modules. The universal spectral flow automorphism $\mathcal{A}$ (3.3) transforms all kinds

of singular vectors back into singular vectors, and chiral states into chiral states. The even spectral flow transformation $\mathcal{U}_1$ (3.7) is very selective, on the contrary, mapping singular vectors of types $|\chi\rangle^{(q)G}_{|\phi\rangle^G}$ and $|\chi\rangle^{(q)G,Q}_{|\phi\rangle^G}$ into singular vectors of types $|\chi\rangle^{(q)Q}_{|\phi\rangle^Q}$ and $|\chi\rangle^{(q)G,Q}_{|\phi\rangle^Q}$ (both), and the other way around for $\mathcal{U}_{-1}$, while mapping all other types of singular vectors into various kinds of states which are not singular vectors. One can consider several other spectral flow mappings, but they are equivalent to the compositions of $\mathcal{A}$, $\mathcal{U}_{\pm1}$ and $\hat{\mathcal{A}}_0$ (the label-exchange operator which interchanges the labels $(1) \leftrightarrow (2)$ of the two sets of topological generators resulting from the two different topological twists).

Third, using $\mathcal{G}_0$, $\mathcal{Q}_0$ and the spectral flow mappings $\mathcal{A}$ and $\mathcal{U}_{\pm1}$, we have derived the family structure of the topological singular vectors, every member of a family being connected to any other member by a chain of transformations. The singular vectors in chiral Verma modules come in families of four with a unique pattern. The families of singular vectors in complete Verma modules follow an infinite number of different patterns, which we have grouped roughly in five main kinds. Two main kinds consist of generic singular vectors, and may contain chiral singular vectors as particular cases, producing the intersection of the two main kinds of families. Another main kind of family contains no-label singular vectors in generic Verma modules, and the remaining two main kinds of families correspond to singular vectors in no-label Verma modules.

Fourth, we have written down the complete families of topological singular vectors in chiral Verma modules until level 3, and we have shown that, starting at level 2, half at least of these singular vectors are subsingular in the complete Verma modules. Namely, we have found that the singular vectors of types $|\chi\rangle^{(1)G}$ and $|\chi\rangle^{(0)Q}$ corresponding to the spectrum $\mathbf{h}^{(1)}_{r,s>2}$ in eq. (4.4), are subsingular vectors in the generic Verma modules $V(|0,\mathbf{h}\rangle^G)$, whereas the singular vectors of types $|\chi\rangle^{(0)G}$ and $|\chi\rangle^{(-1)Q}$ corresponding to the spectrum $\mathbf{h}^{(0)}_{r,s>2}$ in eq. (4.3), are subsingular vectors in the generic Verma modules $V(|0,\mathbf{h}\rangle^Q)$. We conjecture that this result is true at any level.

Fifth, we have derived the spectra of conformal weights Δ and U(1) charges $\mathbf{h}$ corresponding to the generic Verma modules which contain generic and chiral topological singular vectors. From the spectra corresponding to the chiral singular vectors we deduced straightforwardly the spectra corresponding to chiral and antichiral singular vectors of the NS algebra, which were unknown. Regarding the no-label singular vectors in generic Verma modules and the singular vectors in no-label Verma modules we ignore the spectra. The results at level 1 suggest that the no-label singular vectors are very scarce (they only exist for $\mathbf{c} = -3$), and also suggest that the spectrum of $\mathbf{h}$ for no-label Verma modules $V(|0,\, \mathbf{h}\rangle)$ is the "sum" of the spectra corresponding to the generic Verma modules with zero conformal weight $V(|0,\, \mathbf{h}\rangle^G)$ and $V(|0,\, \mathbf{h}\rangle^Q)$.

Sixth, we have analyzed some conditions under which the chains of mappings involving $\mathcal{A}$ and $\mathcal{U}_{\pm 1}$ act inside a Verma module. In generic Verma modules there are many possibilities. In particular we have found a sequence of mappings (sequence B) which has loops, *i.e.* which comes back to the same Verma modules after n steps ($n = 0, 1, 2...$). For singular vectors of the types $|\chi\rangle^{(0)G}_{|\phi\rangle^G}$, $|\chi\rangle^{(-1)Q}_{|\phi\rangle^G}$, $|\chi\rangle^{(0)Q}_{|\phi\rangle^Q}$ and $|\chi\rangle^{(1)G}_{|\phi\rangle^Q}$ these loops connect pairs of singular vectors of the same type at the same level. With this motivation we have analyzed the Dörrzapf conditions (6.1) (originally written for the NS algebra) for the appearance of two linearly independent singular vectors of the same type, at the same level and in the same Verma module, for the case of the loops of mappings of sequence B, finding solutions for every n, starting at level 3. We have constructed singular vectors at level 3 which satisfy these conditions, and belong to the four types mentioned before, and we have checked that the loop of mappings for $n = 0$ transform these singular vectors into singular vectors of the same type, at the same levels, but linearly independent from the initial ones. These facts and several more examples suggest that the loops of mappings made out of $\mathcal{A}$, $\mathcal{U}_{\pm 1}$, $\mathcal{G}_0$ and $\mathcal{Q}_0$, in particular the ones of sequence B, produce pairs of linearly independent singular vectors of types $|\chi\rangle^{(0)G}_{|\phi\rangle^G}$,$|\chi\rangle^{(-1)Q}_{|\phi\rangle^G}$, $|\chi\rangle^{(0)Q}_{|\phi\rangle^Q}$ and $|\chi\rangle^{(1)G}_{|\phi\rangle^Q}$ provided the Dörrzapf conditions are satisfied; otherwise the pairs of singular vectors coincide (up to constants). We conjecture that, in all cases, the two partners of a given Dörrzapf pair are connected by these mappings, *i.e.* they belong to the same family.

Finally, in Appendix A we have given a detailed account of the "cascade effect", in Appendix B we have written down the complete set of topological singular vectors at level 1 for complete Verma modules (twenty-eight different types), and in Appendix C we have shown a specially interesting thirty-eight-member family of generic and chiral singular vectors at levels 3, 4, 5, and 6. This family contains four "Dörrzapf pairs" at level 3, with two partners, corresponding to two different pairs, being chiral singular vectors.

A final comment is that the mappings we have analyzed simplify enormously the construction of topological singular vectors because for a given family it is only necessary to compute one of the singular vectors from scratch. In addition, the singular vectors of the second main kind of families are connected at different levels, and therefore it is possible to construct singular vectors at high levels starting from singular vectors at lower levels. An important observation is that any topological singular vector annihilated by $\mathcal{G}_0$ (*i.e.* $\mathcal{G}_0$-closed or chiral) built on a primary annihilated also by $\mathcal{G}_0$, is transformed into a singular vector of the NS algebra under the untwistings, and the other way around under the twistings. As a consequence the generic singular vectors of types $|\chi\rangle^{(q)G}_{|\phi\rangle^G}$, *i.e.* $\mathcal{G}_0$-closed built on $\mathcal{G}_0$-closed primaries, can be constructed using the corresponding formulae for constructing singular vectors of the NS algebra [8], [9], and performing the topological twists T_{W1} (2.2) and/or T_{W2} (2.3) afterwards. The other types of generic singular vectors

and the chiral singular vectors can be obtained from those ones via the mappings that we have analyzed. For non-generic singular vectors in complete Verma modules and for singular vectors in chiral Verma modules, however, there are no construction formulae‖.

In a forthcoming paper [29] we will analyze thoroughly the implications that the results presented here have on the singular vectors of the NS algebra and on the singular vectors of the R algebra.

The rigorous proof that the cascade effect provides a necessary condition for the existence of a given type of singular vector will be presented in a next publication [22] together with the analysis of the dimensionalities of the spaces associated to every type of singular vector and some related issues.

Acknowledgements

We are indebted to M. Dörrzapf and A. Kent for illuminating discussions. We also thank J.W. van Holten, E.B. Kiritsis, S. Mukhi, A.N. Schellekens and A. Tiemblo for useful comments. Finally, we are very grateful to the (second) referee of Nuclear Physics B for reading this work with so much interest and for many important suggestions to improve its quality.

Appendix A

The "cascade effect" is an algebraic mechanism which consists of the vanishing in cascade of the coefficients of the "would-be" singular vector when imposing the highest weight conditions, alone or in combination with the conditions for BRST or anti-BRST invariance. The vanishing in cascade of the coefficients starts in all cases either with the h.w. conditions $\mathcal{Q}_1|\chi\rangle = 0$ or $\mathcal{G}_1|\chi\rangle = 0$, or with the BRST or anti-BRST invariance conditions $\mathcal{Q}_0|\chi\rangle = 0$ or $\mathcal{G}_0|\chi\rangle = 0$. Once it starts, the cascade effect goes on until the end getting rid of all the coefficients of the would-be singular vectors. The proof of this statement will be presented in a forthcoming publication [22].

For topological singular vectors in chiral Verma modules this effect occurs for all the would-be singular vectors with relative charges $|q| > 1$ and for those with $|q| = 1$ of the types $|\chi\rangle^{(1)Q}$ and $|\chi\rangle^{(-1)G}$. For topological singular vectors in complete Verma modules the cascade effect occurs for all the would-be singular vectors with $|q| > 2$, for most with $|q| = 2$ and for some with $|q| = 1$.

The types of topological singular vectors for which the cascade effect does not take

‖Some attemps have appeared in the literature to write down construction formulae for some types of singular vectors of the Topological algebra, but we find that work very premature and confusing.

place are the ones shown in tables (2.5), (2.6), (2.7) and (2.8), a total of four different types in chiral Verma modules and twenty-nine different types in complete Verma modules, from which one type exists only at level zero ($|\chi\rangle^{(0)G,Q}_{0,|0,\mathbf{h}\rangle} = \mathcal{G}_0\,\mathcal{Q}_0|0,\mathbf{h}\rangle$). The cascade effect not taking place for these types of singular vectors is a necessary, although not sufficient, condition for their existence. However, we know by explicit construction that all these types of singular vectors exist at low levels, even at level 1 (except the type that only exists at level zero). In what follows we will analyze the cascade effect.

Cascade effect in chiral Verma modules

Let us consider topological singular vectors built on chiral primaries $|0,\mathbf{h}\rangle^{G,Q}$. To see how the cascade effect prevents the existence of such singular vectors with relative charge $|q|>1$ let us take a general level-l secondary state with $q=2$. For convenience we will organize the different terms into shells, the first shell consisting of the terms with maximum number of bosonic modes, the second shell containing the terms with maximum number of bosonic modes minus one, and so on. Now by imposing the h.w. condition $\mathcal{Q}_1|\chi\rangle=0$ one deduces easily that the first shell of terms must vanish. The argument goes as follows. For $q=2$ the first shell consists of all possible terms with $l-3$ bosonic modes. Those terms have the structure

$$C_{11}^{pr}\;\; \mathcal{H}_{-1}^{p}\;\; \mathcal{L}_{-1}^{r}\;\; \mathcal{G}_{-2}\;\; \mathcal{G}_{-1}, \quad p+r=l-3 \tag{A.1}$$

where C_{11}^{pr} are the coefficients to be determined. The action of $\mathcal{Q}_1$ on $\mathcal{G}_{-2}$ produces another bosonic mode, since $\{\mathcal{Q}_1,\mathcal{G}_{-2}\}=2\mathcal{L}_{-1}-2\mathcal{H}_{-1}$. Consequently, the terms with $p=0$, $r=l-3$, giving rise to $\mathcal{L}_{-1}^{l-2}\mathcal{G}_{-1}$, and $r=0$, $p=l-3$, giving rise to $\mathcal{H}_{-1}^{l-2}\mathcal{G}_{-1}$ cannot be compensated by any other terms and must vanish (in other words, there are no other terms which will produce $\mathcal{L}_{-1}^{l-2}\mathcal{G}_{-1}$ or $\mathcal{H}_{-1}^{l-2}\mathcal{G}_{-1}$ under the action of $\mathcal{Q}_1$). Applying the same reasoning again we see that the terms with $p=1$, $r=l-4$ and $r=1$, $p=l-4$ must also vanish, due to the vanishing of the terms with $p=0$ and $r=0$. In turn this produces the vanishing of the terms with $p=2$ and $r=2$, and so on. Therefore, *the first shell of terms disappears because of a cascade effect only by imposing the h.w. condition* $\mathcal{Q}_1|\chi\rangle=0$.

The vanishing of the first shell induces, in turn, the vanishing of the second shell, consisting of the terms with $l-4$ bosonic modes, coming in three different structures: $\mathcal{H}_{-2}\,\mathcal{H}_{-1}^{p}\,\mathcal{L}_{-1}^{r}\,\mathcal{G}_{-2}\,\mathcal{G}_{-1}$ and $\mathcal{H}_{-1}^{p}\,\mathcal{L}_{-2}\,\mathcal{L}_{-1}^{r}\,\mathcal{G}_{-2}\,\mathcal{G}_{-1}$, for $p+r=l-5$, and $\mathcal{H}_{-1}^{p}\,\mathcal{L}_{-1}^{r}\,\mathcal{G}_{-3}\,\mathcal{G}_{-1}$ for $p+r=l-4$. By imposing $\mathcal{H}_1|\chi\rangle=0$ and again $\mathcal{Q}_1|\chi\rangle=0$, one finds three inequivalent blocks containing the terms which compensate each other in such a way that all the coefficients must vanish. The vanishing of the second shell of terms induces, in turn, the vanishing of the third shell, with $l-5$ bosonic modes, and so on, as the reader can verify. Hence there are no topological singular vectors with relative charge $|q|=2$ in

chiral Verma modules (the same analysis can be repeated for the case $q = -2$, giving the same results, just by interchanging $\mathcal{Q} \leftrightarrow \mathcal{G}$ everywhere).

The same mechanism takes place for the would-be singular vectors with $q > 2$ (and $q < -2$ as a consequence) because the first shell of terms with maximum number of bosonic modes has the unique structure $\mathcal{H}_{-1}^p\ \mathcal{L}_{-1}^r .... \mathcal{G}_{-3}\, \mathcal{G}_{-2}\, \mathcal{G}_{-1}$, in chiral Verma modules, where the number of $\mathcal{G}$ modes is equal to q. The action of $\mathcal{Q}_1$ on these terms produces a new bosonic mode when hitting a $\mathcal{G}$-mode other than $\mathcal{G}_{-1}$. As a result the terms with $p = 0$ and $r = 0$ cannot be compensated by any other terms and must vanish, giving rise to the vanishing of the complete shell of terms in a cascade effect, as in the case $q = 2$.

In the case $|q| = 1$, for secondary states of types $|\chi\rangle^{(1)G}$ and $|\chi\rangle^{(-1)Q}$ there is no cascade effect getting rid of the terms. However, there is such effect for the would-be singular vectors of types $|\chi\rangle^{(1)Q}$ and $|\chi\rangle^{(-1)G}$. For $q = 1$ the action of $\mathcal{Q}_1$ on $\mathcal{G}_{-1}$, in the first shell of terms $\mathcal{H}_{-1}^p\ \mathcal{L}_{-1}^r\ \mathcal{G}_{-1}$, does not create a new bosonic mode but a number: $\mathcal{H}_{-1}^p\ \mathcal{L}_{-1}^r\ \mathcal{L}_0 \rightarrow \Delta \mathcal{H}_{-1}^p\ \mathcal{L}_{-1}^r$ and $\mathcal{H}_{-1}^p\ \mathcal{L}_{-1}^r\ \mathcal{H}_0 \rightarrow \mathbf{h}\mathcal{H}_{-1}^p\ \mathcal{L}_{-1}^r$, where $\mathbf{h}$ and Δ are the U(1) charge and the conformal weight of the topological primary state on which the secondary is built (for topological chiral primaries $\Delta = 0$). However, if one imposes BRST-invariance on the $q = 1$ would-be singular vector, *i.e.* $\mathcal{Q}_0|\chi\rangle = 0$, the vanishing of the terms starts again since $\mathcal{Q}_0$ does indeed create a bosonic mode ($\mathcal{L}_{-1}$) when acting on $\mathcal{G}_{-1}$. The first shell of terms, with $l - 1$ bosonic modes vanishes in this way. Therefore there is a cascade effect getting rid of the terms of the would-be singular vectors of the types $|\chi\rangle^{(1)Q}$ and $|\chi\rangle^{(-1)G}$ (for the latter by interchanging $\mathcal{Q} \leftrightarrow \mathcal{G}$ everywhere in the previous expressions).

Let us consider finally the case $q = 0$. Since the first shell $\mathcal{H}_{-1}^p\ \mathcal{L}_{-1}^r$ contains only bosonic modes, it is not possible to create any other bosonic mode by acting with the positive modes or with $\mathcal{Q}_0$ or $\mathcal{G}_0$. Therefore the cascade effect does not take place in the case $q = 0$.

Cascade effect in complete Verma modules

When the topological primary is $\mathcal{Q}_0$-closed $|\Delta, \mathbf{h}\rangle^Q$ or no-label $|\Delta, \mathbf{h}\rangle$, the $\mathcal{G}_0$-modes also contribute to build the Verma module, unlike in the chiral case. As a result, for $q > 0$ the first shell of terms with maximum number of bosonic modes contains one $\mathcal{G}_0$-mode. Similarly, when the topological primary is $\mathcal{G}_0$-closed $|\Delta, \mathbf{h}\rangle^G$ or no-label, the $\mathcal{Q}_0$-modes contribute to build the Verma module, and for $q < 0$ the first shell contains one $\mathcal{Q}_0$-mode.

The resulting structures for the first shells of terms, depending on the type of primary state, are therefore the following. For $q > 2$ one has $\mathcal{H}_{-1}^p\ \mathcal{L}_{-1}^r .... \mathcal{G}_{-3}\, \mathcal{G}_{-2}\, \mathcal{G}_{-1}\, |\Delta, \mathbf{h}\rangle^G$, $\mathcal{H}_{-1}^p\ \mathcal{L}_{-1}^r .... \mathcal{G}_{-2}\, \mathcal{G}_{-1}\, \mathcal{G}_0\, |\Delta, \mathbf{h}\rangle^Q$ and $\mathcal{H}_{-1}^p\ \mathcal{L}_{-1}^r .... \mathcal{G}_{-2}\, \mathcal{G}_{-1}\, \mathcal{G}_0\, |0, \mathbf{h}\rangle$. The cascade

effect occurs in all three cases simply by imposing $\mathcal{Q}_1|\chi\rangle = 0$. For $q < -2$ one finds the same result by interchanging $\mathcal{Q} \leftrightarrow \mathcal{G}$ everywhere. Hence the cascade effect always takes place for the would-be topological singular vectors with $|q| > 2$.

For $q = 2$ the first shells are $\mathcal{H}^p_{-1}\, \mathcal{L}^r_{-1}\, \mathcal{G}_{-2}\, \mathcal{G}_{-1}\, |\Delta, \mathbf{h}\rangle^G$, $\mathcal{H}^p_{-1}\, \mathcal{L}^r_{-1}\, \mathcal{G}_{-1}\, \mathcal{G}_0\, |\Delta, \mathbf{h}\rangle^Q$ and $\mathcal{H}^p_{-1}\, \mathcal{L}^r_{-1}\, \mathcal{G}_{-1}\, \mathcal{G}_0\, |0, \mathbf{h}\rangle$. The cascade effect starts in the first case by imposing $\mathcal{Q}_1|\chi\rangle = 0$ whereas it starts in the second and third cases only by imposing $\mathcal{Q}_0|\chi\rangle = 0$. Therefore for $q = 2$ the cascade effect takes place for the would-be singular vectors of the types $|\chi\rangle^{(2)G}_{|\phi\rangle^G}$, $|\chi\rangle^{(2)G,Q}_{|\phi\rangle^G}$, $|\chi\rangle^{(2)Q}_{|\phi\rangle^G}$, $|\chi\rangle^{(2)}_{|\phi\rangle^G}$, $|\chi\rangle^{(2)Q}_{|\phi\rangle^Q}$, $|\chi\rangle^{(2)G,Q}_{|\phi\rangle^Q}$ and $|\chi\rangle^{(2)Q}_{|0,\mathbf{h}\rangle}$. One finds similar results for $q = -2$, interchanging $\mathcal{Q} \leftrightarrow \mathcal{G}$ in all these types.

For $q = 1$ the first shells of terms are $\mathcal{H}^p_{-1}\, \mathcal{L}^r_{-1}\, \mathcal{G}_{-1}\, |\Delta, \mathbf{h}\rangle^G$, $\mathcal{H}^p_{-1}\, \mathcal{L}^r_{-1}\, \mathcal{G}_0\, |\Delta, \mathbf{h}\rangle^Q$ and $\mathcal{H}^p_{-1}\, \mathcal{L}^r_{-1}\, \mathcal{G}_0\, |0, \mathbf{h}\rangle$. The cascade effect occurs only in the first case, by imposing $\mathcal{Q}_0|\chi\rangle = 0$; that is, only for the would-be singular vectors of the types $|\chi\rangle^{(1)Q}_{|\phi\rangle^G}$ and $|\chi\rangle^{(1)G,Q}_{|\phi\rangle^G}$. One finds similar results for $q = -1$, interchanging $\mathcal{Q} \leftrightarrow \mathcal{G}$.

For $q = 0$ the cascade effect does not take place, obviously, like in the case of chiral Verma modules.

The cascade effect also prevents, indirectly, the existence of no-label singular vectors of types $|\chi\rangle^{(2)}_{|\phi\rangle^Q}$, $|\chi\rangle^{(-2)}_{|\phi\rangle^G}$, $|\chi\rangle^{(1)}_{|\phi\rangle^G}$ and $|\chi\rangle^{(-1)}_{|\phi\rangle^Q}$. The reason is that acting with $\mathcal{G}_0$ or $\mathcal{Q}_0$ on these singular vectors one obtains $\mathcal{G}_0$-closed or $\mathcal{Q}_0$-closed singular vectors forbidden by the cascade effect. For example, $\mathcal{G}_0|\chi\rangle^{(1)}_{|\phi\rangle^G} = |\chi\rangle^{(2)G}_{|\phi\rangle^G}$, but the latter type does not exist, so that $|\chi\rangle^{(1)}_{|\phi\rangle^G}$ must be $\mathcal{G}_0$-closed, *i.e.* of type $|\chi\rangle^{(1)G}_{|\phi\rangle^G}$ instead**.

Appendix B

Let us write down the complete set of topological singular vectors at level 1 in complete Verma modules.

Generic singular vectors

First let us consider the four types of singular vectors associated to the skeleton-family (5.1); that is, the generic singular vectors of the first main kind of families. The uncharged $\mathcal{G}_0$-closed topological singular vectors at level 1, built on $\mathcal{G}_0$-closed primaries, are given by

$$|\chi\rangle^{(0)G}_{1,|\Delta,\,\mathbf{h}\rangle^G} = ((\mathbf{h}-1)\mathcal{L}_{-1} + (\mathbf{h}-1-2\Delta)\mathcal{H}_{-1} + \mathcal{G}_{-1}\mathcal{Q}_0)|\Delta,\,\mathbf{h}\rangle^G, \tag{B.1}$$

where Δ and $\mathbf{h}$ are related by the quadratic vanishing surface $f_{1,2}(\Delta,\,\mathbf{h}) = 0$, eq. (5.10), as $2\Delta t = -\mathbf{h}^2 + t\mathbf{h} + 1 - t$, with $t = (3-\mathbf{c})/3$. By using $\mathcal{A}$ (3.3) one obtains $|\chi\rangle^{(0)Q}_{1,|\Delta,\,\mathbf{h}'\rangle^Q} =$

**We thank M. Dörrzapf for pointing out this to us.

$\mathcal{A}\,|\chi\rangle^{(0)G}_{1,|\Delta,\,\mathbf{h}\rangle^G}$, with $\mathbf{h}'=-\mathbf{h}-\frac{\mathbf{c}}{3}$. That is,

$$|\chi\rangle^{(0)Q}_{1,|\Delta,\,\mathbf{h}'\rangle^Q}=((-\mathbf{h}'-\frac{\mathbf{c}}{3}-1)\mathcal{L}_{-1}+2\Delta\mathcal{H}_{-1}+\mathcal{Q}_{-1}\mathcal{G}_0)|\Delta,\,\mathbf{h}'\rangle^Q, \tag{B.2}$$

with Δ and $\mathbf{h}'$ related by $2\Delta t=-\mathbf{h}'^2+t\mathbf{h}'-2\mathbf{h}'$. Now acting with $\mathcal{Q}_0$ and $\mathcal{G}_0$ on $|\chi\rangle^{(0)G}_{1,|\Delta,\,\mathbf{h}\rangle^G}$ and $|\chi\rangle^{(0)Q}_{1,|\Delta,\,\mathbf{h}'\rangle^Q}$, respectively, one obtains

$$|\chi\rangle^{(-1)Q}_{1,|\Delta,\,\mathbf{h}\rangle^G}=(\frac{3-\mathbf{c}}{3(\mathbf{h}-1)}\mathcal{L}_{-1}\mathcal{Q}_0+\mathcal{H}_{-1}\mathcal{Q}_0+\mathcal{Q}_{-1})|\Delta,\,\mathbf{h}\rangle^G, \tag{B.3}$$

$$|\chi\rangle^{(1)G}_{1,|\Delta,\,\mathbf{h}'\rangle^Q}=(\frac{\mathbf{c}-3}{3\mathbf{h}'+\mathbf{c}+3}\mathcal{L}_{-1}\mathcal{G}_0-\frac{6+3\mathbf{h}'}{3\mathbf{h}'+\mathbf{c}+3}\mathcal{H}_{-1}\mathcal{G}_0+\mathcal{G}_{-1})|\Delta,\,\mathbf{h}'\rangle^Q. \tag{B.4}$$

Therefore for given values of Δ and $\mathbf{h}$, and setting $\mathbf{h}'=-\mathbf{h}-\frac{\mathbf{c}}{3}$, one gets a four-member subfamily attached to another similar four-member subfamily with $\hat{\Delta}=\Delta-\mathbf{h}$ and $\hat{\mathbf{h}}=-\mathbf{h}$, as one can see in diagram (5.1). Whether or not there are further attachements depends on the values of Δ, $\mathbf{h}$ and $\mathbf{c}$.

Let us consider now the eight types of singular vectors associated to the skeleton-family (5.2); that is, the generic singular vectors of the second main kind of families. The singular vectors of types $|\chi\rangle^{(1)G}_{1,|\Delta,\,\mathbf{h}\rangle^G}$ and $|\chi\rangle^{(-1)Q}_{1,|\Delta,\,\mathbf{h}'\rangle^Q}=\mathcal{A}\,|\chi\rangle^{(1)G}_{1,|\Delta,\,\mathbf{h}\rangle^G}$ are given by

$$|\chi\rangle^{(1)G}_{1,|\Delta,\,\mathbf{h}\rangle^G}=\mathcal{G}_{-1}\,|\Delta,\,\mathbf{h}\rangle^G,\qquad |\chi\rangle^{(-1)Q}_{1,|\Delta,\,\mathbf{h}'\rangle^Q}=\mathcal{Q}_{-1}\,|\Delta,\,\mathbf{h}'\rangle^Q\,, \tag{B.5}$$

with $\mathbf{h}'=-\mathbf{h}-\frac{\mathbf{c}}{3}$ and $\Delta=\mathbf{h}$, corresponding to the vanishing plane $g_{1/2}(\Delta,\mathbf{h})=0$, eq. (5.11). Acting with $\mathcal{Q}_0$ and $\mathcal{G}_0$ on these vectors one completes the four-member subdiagram associated to $|\chi\rangle^{(1)G}_{1,|\Delta,\,\mathbf{h}\rangle^G}$, obtaining

$$|\chi\rangle^{(0)Q}_{1,|\Delta,\,\mathbf{h}\rangle^G}=\mathcal{Q}_0\mathcal{G}_{-1}\,|\Delta,\,\mathbf{h}\rangle^G,\qquad |\chi\rangle^{(0)G}_{1,|\Delta,\,\mathbf{h}'\rangle^Q}=\mathcal{G}_0\mathcal{Q}_{-1}\,|\Delta,\,\mathbf{h}'\rangle^Q\,. \tag{B.6}$$

This subdiagram is attached to a chiral couple at level zero, and more attachements may be possible (following sequence B) depending on the values of $\mathbf{h}$ and $\mathbf{c}$.

The remaining four types of generic singular vectors at level 1, which also provide a four-member subdiagram, attached to a four-member subdiagram at level 2, are:

$$|\chi\rangle^{(2)G}_{1,|\Delta,\,\mathbf{h}'\rangle^Q}=\mathcal{G}_{-1}\mathcal{G}_0\,|\Delta,\,\mathbf{h}'\rangle^Q,\qquad |\chi\rangle^{(-2)Q}_{1,|\Delta,\,\mathbf{h}\rangle^G}=\mathcal{Q}_{-1}\mathcal{Q}_0\,|\Delta,\,\mathbf{h}\rangle^G, \tag{B.7}$$

$$|\chi\rangle^{(1)Q}_{1,|\Delta,\,\mathbf{h}'\rangle^Q}=(\mathcal{L}_{-1}\mathcal{G}_0-\Delta\mathcal{G}_{-1})|\Delta,\,\mathbf{h}'\rangle^Q, \tag{B.8}$$

$$|\chi\rangle^{(-1)G}_{1,|\Delta,\,\mathbf{h}\rangle^G} = (\mathcal{L}_{-1}\mathcal{Q}_0 + \mathcal{H}_{-1}\mathcal{Q}_0 - \Delta\mathcal{Q}_{-1})|\Delta,\,\mathbf{h}\rangle^G, \tag{B.9}$$

where $\mathbf{h}' = -\mathbf{h} - \frac{\mathbf{c}}{3}$ and $\mathbf{h}$ and Δ are related by the vanishing plane $g_{-3/2}(\Delta, \mathbf{h}) = 0$ as $\Delta + \mathbf{h} - 1 + \mathbf{c}/3 = 0$. As before, it may be further attachements depending on the values of Δ, $\mathbf{h}$ and $\mathbf{c}$.

As is explained in section 2, all the singular vectors built on primaries of type $|\Delta, \mathbf{h}\rangle^G$ are equivalent to singular vectors built on primaries of type $|\Delta, \mathbf{h}\rangle^Q$ (with different assignements of the U(1) charges q and $\mathbf{h}$), provided $\Delta \neq 0$, as the reader can easily verify.

Chiral singular vectors

Chiral singular vectors only exist in the types $|\chi\rangle^{(0)G,Q}_{|\phi\rangle^G}$, $|\chi\rangle^{(0)G,Q}_{|\phi\rangle^Q}$, $|\chi\rangle^{(-1)G,Q}_{|\phi\rangle^G}$ and $|\chi\rangle^{(1)G,Q}_{|\phi\rangle^Q}$, and at level 1 they require $\Delta = -1$. The uncharged chiral singular vectors are given by

$$|\chi\rangle^{(0)G,Q}_{1,|-1,-1\rangle^G} = (-2\mathcal{L}_{-1} + \mathcal{G}_{-1}\mathcal{Q}_0)|-1,-1\rangle^G, \tag{B.10}$$

$$|\chi\rangle^{(0)G,Q}_{1,|-1,\,\frac{3-\mathbf{c}}{3}\rangle^Q} = (-2\mathcal{L}_{-1} - 2\mathcal{H}_{-1} + \mathcal{Q}_{-1}\mathcal{G}_0)|-1,\,\frac{3-\mathbf{c}}{3}\rangle^Q, \tag{B.11}$$

and the charged chiral singular vectors are given by

$$|\chi\rangle^{(-1)G,Q}_{1,\,|-1,\,\frac{6-\mathbf{c}}{3}\rangle^G} = (\mathcal{L}_{-1}\mathcal{Q}_0 + \mathcal{H}_{-1}\mathcal{Q}_0 + \mathcal{Q}_{-1})|-1,\,\frac{6-\mathbf{c}}{3}\rangle^G, \tag{B.12}$$

$$|\chi\rangle^{(1)G,Q}_{1,|-1,-2\rangle^Q} = (\mathcal{L}_{-1}\mathcal{G}_0 + \mathcal{G}_{-1})|-1,-2\rangle^Q. \tag{B.13}$$

Observe that the chiral singular vectors are the intersection of the corresponding $\mathcal{G}_0$-closed and $\mathcal{Q}_0$-closed singular vectors in each case. Observe also that $|\chi\rangle^{(0)G,Q}_{1,|-1,-1\rangle^G}$ and $|\chi\rangle^{(1)G,Q}_{1,|-1,-2\rangle^Q}$ on the one side, and $|\chi\rangle^{(0)G,Q}_{1,|-1,\,\frac{3-\mathbf{c}}{3}\rangle^Q}$ and $|\chi\rangle^{(-1)G,Q}_{1,|-1,\frac{6-\mathbf{c}}{3}\rangle^G}$ on the other side, are equivalent just by expressing $|-1,-1\rangle^G = \mathcal{G}_0|-1,-2\rangle^Q$ and $|-1,\,\frac{3-\mathbf{c}}{3}\rangle^Q = \mathcal{Q}_0|-1,\,\frac{6-\mathbf{c}}{3}\rangle^G$.

No-label singular vectors

The no-label singular vectors, *i.e.* which cannot be expressed as linear combinations of $\mathcal{G}_0$-closed and $\mathcal{Q}_0$-closed singular vectors, exist only in the types $|\chi\rangle^{(1)}_{|\phi\rangle^Q}$ and $|\chi\rangle^{(-1)}_{|\phi\rangle^G}$, and the equivalent types $|\chi\rangle^{(0)}_{|\phi\rangle^G}$, and $|\chi\rangle^{(0)}_{|\phi\rangle^Q}$. At level 1 they require $\Delta = -1$. One finds solutions only for $\mathbf{c} = -3$. They are:

$$|\chi\rangle^{(1)}_{1,|-1,-2\rangle^Q} = (\mathcal{L}_{-1}\mathcal{G}_0 - \mathcal{H}_{-1}\mathcal{G}_0)|-1,-2\rangle^Q, \tag{B.14}$$

$$|\chi\rangle^{(-1)}_{1,\,|-1,\,3\rangle^G} = (\mathcal{L}_{-1}\mathcal{Q}_0 + 2\mathcal{H}_{-1}\mathcal{Q}_0)|-1,\,3\rangle^G, \tag{B.15}$$

$$|\chi\rangle^{(0)}_{1,|-1,-1\rangle^G} = (\mathcal{L}_{-1} - \mathcal{H}_{-1})|-1,-1\rangle^G, \tag{B.16}$$

$$|\chi\rangle^{(0)}_{1,|-1,\,2\rangle^Q} = (\mathcal{L}_{-1} + 2\mathcal{H}_{-1})|-1,\,2\rangle^Q. \tag{B.17}$$

The conformal weight of these singular vectors is zero. Thus acting with $\mathcal{G}_0$ and $\mathcal{Q}_0$ one obtains $\mathcal{G}_0$-closed and $\mathcal{Q}_0$-closed secondary singular vectors, respectively.

Singular vectors in no-label Verma modules

Now let us consider no-label Verma modules built on no-label primaries $|0,\mathbf{h}\rangle$. The corresponding singular vectors are grouped in two different kinds of families, as diagram (5.7) shows. At level 1 one finds one solution for the families on the right-hand side of the diagram:

$$|\chi\rangle^{(2)G}_{1,|0,-1\rangle} = \mathcal{G}_{-1}\mathcal{G}_0\,|0,-1\rangle, \qquad |\chi\rangle^{(-2)Q}_{1,|0,\frac{3-\mathbf{c}}{3}\rangle} = \mathcal{Q}_{-1}\mathcal{Q}_0\,|0,\frac{3-\mathbf{c}}{3}\rangle, \tag{B.18}$$

$$|\chi\rangle^{(1)Q}_{1,|0,-1\rangle} = \mathcal{Q}_0\mathcal{G}_{-1}\mathcal{G}_0\,|0,-1\rangle, \qquad |\chi\rangle^{(-1)G}_{1,|0,\frac{3-\mathbf{c}}{3}\rangle} = \mathcal{G}_0\mathcal{Q}_{-1}\mathcal{Q}_0\,|0,\frac{3-\mathbf{c}}{3}\rangle, \tag{B.19}$$

and two solutions for the families on the left-hand side of the diagram (which intersect for $\mathbf{c}=-3$):

$$|\chi\rangle^{(1)G}_{1,|0,0\rangle} = \mathcal{G}_{-1}\mathcal{G}_0\mathcal{Q}_0\,|0,0\rangle, \qquad |\chi\rangle^{(-1)Q}_{1,|0,-\frac{\mathbf{c}}{3}\rangle} = \mathcal{Q}_{-1}\mathcal{Q}_0\mathcal{G}_0\,|0,-\frac{\mathbf{c}}{3}\rangle, \tag{B.20}$$

$$|\chi\rangle^{(0)Q}_{1,|0,0\rangle} = \mathcal{L}_{-1}\mathcal{G}_0\mathcal{Q}_0\,|0,0\rangle, \qquad |\chi\rangle^{(0)G}_{1,|0,-\frac{\mathbf{c}}{3}\rangle} = (\mathcal{L}_{-1}\mathcal{G}_0\mathcal{Q}_0 + \mathcal{H}_{-1}\mathcal{G}_0\mathcal{Q}_0)|0,-\frac{\mathbf{c}}{3}\rangle, \tag{B.21}$$

and

$$|\chi\rangle^{(1)G}_{1,|0,-\frac{\mathbf{c}+3}{3}\rangle} = (\frac{\mathbf{c}+3}{3}\mathcal{L}_{-1}\mathcal{G}_0 + \frac{\mathbf{c}+3}{3}\mathcal{H}_{-1}\mathcal{G}_0 + \mathcal{G}_{-1}\mathcal{G}_0\mathcal{Q}_0)|0,-\frac{\mathbf{c}+3}{3}\rangle, \tag{B.22}$$

$$|\chi\rangle^{(-1)Q}_{1,|0,1\rangle} = (\frac{\mathbf{c}+3}{3}\mathcal{L}_{-1}\mathcal{Q}_0 + \mathcal{Q}_{-1}\mathcal{Q}_0\mathcal{G}_0)|0,1\rangle, \tag{B.23}$$

$$|\chi\rangle^{(0)Q}_{1,|0,-\frac{\mathbf{c}+3}{3}\rangle} = ((\mathbf{c}-3)\mathcal{L}_{-1}\mathcal{Q}_0\mathcal{G}_0 + (\mathbf{c}+3)\mathcal{H}_{-1}\mathcal{Q}_0\mathcal{G}_0 + (\mathbf{c}+3)\mathcal{Q}_{-1}\mathcal{G}_0)|0,-\frac{\mathbf{c}+3}{3}\rangle, \tag{B.24}$$

$$|\chi\rangle^{(0)G}_{1,|0,1\rangle} = (\frac{3-\mathbf{c}}{6}\mathcal{L}_{-1}\mathcal{G}_0\mathcal{Q}_0 + \mathcal{H}_{-1}\mathcal{G}_0\mathcal{Q}_0 - \frac{\mathbf{c}+3}{6}\mathcal{G}_{-1}\mathcal{Q}_0)|0,1\rangle. \tag{B.25}$$

The spectrum of $\mathbf{h}$ that we have found at level 1 for no-label Verma modules $V(|0,\mathbf{h}\rangle)$ is the "sum" of the spectra of $\mathbf{h}$ at level 1 corresponding to the generic Verma modules with zero conformal weight $V(|0,\mathbf{h}\rangle^G)$ and $V(|0,\mathbf{h}\rangle^Q)$, as the interested reader can verify.

Loop of sequence B

To finish, let us show that the loop of sequence B for $\mathbf{h} = \frac{3-\mathbf{c}}{6} = t/2$ transforms a singular vector of type $|\chi\rangle^{(0)G}_{1,|\Delta,\mathbf{h}\rangle^G}$, with $\Delta \neq 0$, back to itsef. This result is to be expected because at level 1 the Dörrzapf conditions (6.1) cannot be satisfied.

For $\Delta \neq 0$ the subfamily associated to $|\chi\rangle^{(0)G}_{1,|\Delta,\mathbf{h}\rangle^G}$, given by (B.1), can be expressed as shown in diagram (5.3). In order to obtain $|\chi\rangle^{(0)G}_{1,|\Delta,-\mathbf{h}-\mathbf{c}/3+1\rangle^G} = \mathcal{G}_0\,\mathcal{A}|\chi\rangle^{(0)G}_{1,|\Delta,\mathbf{h}\rangle^G}$ one has to apply $\mathcal{G}_0\,\mathcal{A}$ first, then one has to express the resulting $\mathcal{Q}_0$-closed primary as $|\Delta,-\mathbf{h}-\mathbf{c}/3\rangle^Q = \mathcal{Q}_0\,|\Delta,-\mathbf{h}-\mathbf{c}/3+1\rangle^G = \mathcal{Q}_0\,|\Delta,-\mathbf{h}+t\rangle^G$, resulting finally in the expression

$$|\chi\rangle^{(0)G}_{1,|\Delta,-\mathbf{h}+t\rangle^G} = ((\mathbf{h}+1)2\Delta\mathcal{L}_{-1} + 4(\Delta^2+\Delta)\mathcal{H}_{-1} + (\mathbf{h}-1-2\Delta)\mathcal{G}_{-1}\mathcal{Q}_0)|\Delta,-\mathbf{h}+t\rangle^G. \tag{B.26}$$

Now setting $\mathbf{h} = t/2$ in both vectors, (B.1) and (B.26), and taking into account that $\Delta = (t-2)^2/8t$, given by $f_{1,2}(\Delta, t/2) = 0$ in (5.10), one arrives at the same expression:

$$|\chi\rangle^{(0)G}_{1,|\Delta,t/2\rangle^G} = (\frac{t-2}{2}\mathcal{L}_{-1} + \frac{t^2-4}{4t}\mathcal{H}_{-1} + \mathcal{G}_{-1}\mathcal{Q}_0)|\Delta,t/2\rangle^G \tag{B.27}$$

Therefore, the two uncharged singular vectors $|\chi\rangle^{(0)G}_{1,|\Delta,\mathbf{h}\rangle^G}$ and $\mathcal{G}_0\,\mathcal{A}|\chi\rangle^{(0)G}_{1,|\Delta,\mathbf{h}\rangle^G}$ coincide for $\mathbf{h} = \frac{3-\mathbf{c}}{6} = t/2$, provided $\Delta \neq 0$.

To see that the mapping given by $\mathcal{G}_0\,\mathcal{A}$ is far from being the identity, let us repeat the same procedure for a general $\mathcal{G}_0$-closed uncharged secondary state at level 1 instead of a singular vector, *i.e.* for the state

$$|\gamma\rangle^{(0)G}_{1,|\Delta,\mathbf{h}\rangle^G} = (\alpha\mathcal{L}_{-1} + \beta\mathcal{H}_{-1} + \mathcal{G}_{-1}\mathcal{Q}_0)|\Delta,\,\mathbf{h}\rangle^G\ , \tag{B.28}$$

with $\alpha - \beta = 2\Delta$. One finds that $\mathcal{G}_0\,\mathcal{A}$ maps the secondary state back to itself, for $\mathbf{h} = t/2$, only if $\beta^2 = 4\Delta(\Delta+1)$. This is in fact the case for the singular vector (B.1) for that particular value of $\mathbf{h}$.

Appendix C

Here we show a specially interesting thirty-eight-member family of generic and chiral topological singular vectors at levels 3, 4, 5, and 6, for $\mathbf{c} = 9$ ($t = -2$). It contains four Dörrzapf pairs, at level 3, and three chiral couples which produce the intersection of the singular vectors associated to the skeleton-family (5.1) (first kind of generic families) with the singular vectors associated to the skeleton-family (5.2) (second kind of generic families). Due to the length of this family we write down explicitely only the four Dörrzapf pairs, displayed in diagram (C.1). The upper row and the lower row of the diagram are connected by $\mathcal{U}_1$, as indicated by the arrow on top, since $\mathcal{U}_1\mathcal{A}\,\mathcal{U}_1\mathcal{A} = \mathbf{I}$

$$
\begin{array}{ccccccc}
\mathcal{U}_1 \uparrow & & & & & & \\
|\chi\rangle^{(0)G}_{3,\,|-4,-1\rangle^G} & \xrightarrow{\mathcal{Q}_0} & |\hat{\chi}\rangle^{(0)Q}_{3,\,|-4,-2\rangle^Q} & \xrightarrow{\mathcal{U}_{-1}} & |\chi\rangle^{(0)G}_{3,\,|-3,\,1\rangle^G} & \xrightarrow{\mathcal{Q}_0} & \\
\mathcal{A} \updownarrow & & \updownarrow \mathcal{A} & & \updownarrow \mathcal{A} & & \\
|\chi\rangle^{(0)Q}_{3,\,|-4,-2\rangle^Q} & \xrightarrow{\mathcal{G}_0} & |\hat{\chi}\rangle^{(0)G}_{3,\,|-4,-1\rangle^G} & \xrightarrow{\mathcal{U}_{1}} & |\chi\rangle^{(0)Q}_{3,\,|-3,-4\rangle^Q} & \xrightarrow{\mathcal{G}_0} & \\
\mathcal{U}_1 \uparrow & & & & & & \\
|\chi\rangle^{(0)G,Q}_{3,\,|-3,\,1\rangle^G} & & & & & & \\
\mathcal{A} \updownarrow & & & & & & \\
|\chi\rangle^{(0)G,Q}_{3,\,|-3,-4\rangle^Q} & & & & & &
\end{array}
\tag{C.1}
$$

The uncharged chiral couple of singular vectors $|\chi\rangle^{(0)G,Q}_{3,\,|-3,\,1\rangle^G}$ and $|\chi\rangle^{(0)G,Q}_{3,\,|-3,-4\rangle^Q}$,

$$
\begin{aligned}
|\chi\rangle^{(0)G,Q}_{3,\,|-3,\,1\rangle^G} = \mathcal{Q}_0\,(&3\mathcal{L}_{-2}\mathcal{G}_{-1} + \mathcal{L}^2_{-1}\mathcal{G}_{-1} - \mathcal{H}_{-2}\mathcal{G}_{-1} + 2\mathcal{H}^2_{-1}\mathcal{G}_{-1} - 3\mathcal{L}_{-1}\mathcal{H}_{-1}\mathcal{G}_{-1} + \\
&4\mathcal{L}_{-1}\mathcal{G}_{-2} - 8\mathcal{H}_{-1}\mathcal{G}_{-2} + 12\mathcal{G}_{-3})|-3,1\rangle^G,
\end{aligned}
$$

$$
\begin{aligned}
|\chi\rangle^{(0)G,Q}_{3,\,|-3,-4\rangle^Q} = \mathcal{G}_0\,(&3\mathcal{L}_{-2}\mathcal{Q}_{-1} + \mathcal{L}^2_{-1}\mathcal{Q}_{-1} + 6\mathcal{H}_{-2}\mathcal{Q}_{-1} + 6\mathcal{H}^2_{-1}\mathcal{Q}_{-1} + 5\mathcal{L}_{-1}\mathcal{H}_{-1}\mathcal{Q}_{-1} + \\
&4\mathcal{L}_{-1}\mathcal{Q}_{-2} + 12\mathcal{H}_{-1}\mathcal{Q}_{-2} + 12\mathcal{Q}_{-3})|-3,-4\rangle^Q\,,
\end{aligned}
$$

is connected by $\mathcal{U}_{\pm1}$ to the pair of singular vectors $|\chi\rangle^{(0)Q}_{3,|-4,-2\rangle^Q}$ and $|\chi\rangle^{(0)G}_{3,|-4,-1\rangle^G}$. Acting further with $\mathcal{G}_0$ and $\mathcal{Q}_0$ one obtains the four-member subfamily

$$|\chi\rangle^{(0)Q}_{3,|-4,-2\rangle^Q} = (3\mathcal{L}_{-2}\mathcal{Q}_{-1}\mathcal{G}_0 + \mathcal{L}^2_{-1}\mathcal{Q}_{-1}\mathcal{G}_0 + \mathcal{H}_{-2}\mathcal{Q}_{-1}\mathcal{G}_0 - \mathcal{L}_{-1}\mathcal{H}_{-1}\mathcal{Q}_{-1}\mathcal{G}_0 + 4\mathcal{L}_{-1}\mathcal{Q}_{-1}\mathcal{G}_{-1} + \\ -3\mathcal{L}_{-1}\mathcal{Q}_{-2}\mathcal{G}_0 - 4\mathcal{H}_{-1}\mathcal{Q}_{-1}\mathcal{G}_{-1} + \mathcal{H}_{-1}\mathcal{Q}_{-2}\mathcal{G}_0 - 8\mathcal{Q}_{-2}\mathcal{G}_{-1} + 12\mathcal{Q}_{-1}\mathcal{G}_{-2} + \mathcal{Q}_{-3}\mathcal{G}_0)|-4,-2\rangle^Q,$$

$$|\chi\rangle^{(0)G}_{3,|-4,-1\rangle^G} = (3\mathcal{L}_{-2}\mathcal{G}_{-1}\mathcal{Q}_0 + \mathcal{L}^2_{-1}\mathcal{G}_{-1}\mathcal{Q}_0 + 4\mathcal{H}_{-2}\mathcal{G}_{-1}\mathcal{Q}_0 + 2\mathcal{H}^2_{-1}\mathcal{G}_{-1}\mathcal{Q}_0 + \\ 3\mathcal{L}_{-1}\mathcal{H}_{-1}\mathcal{G}_{-1}\mathcal{Q}_0 + 4\mathcal{L}_{-1}\mathcal{G}_{-1}\mathcal{Q}_{-1} + 8\mathcal{H}_{-1}\mathcal{G}_{-1}\mathcal{Q}_{-1} - 3\mathcal{L}_{-1}\mathcal{G}_{-2}\mathcal{Q}_0 + \\ -4\mathcal{H}_{-1}\mathcal{G}_{-2}\mathcal{Q}_0 - 8\mathcal{G}_{-2}\mathcal{Q}_{-1} + 12\mathcal{G}_{-1}\mathcal{Q}_{-2} + \mathcal{G}_{-3}\mathcal{Q}_0)|-4,-1\rangle^G,$$

$$|\chi\rangle^{(-1)Q}_{3,|-4,-1\rangle^G} = |\hat{\chi}\rangle^{(0)Q}_{3,|-4,-2\rangle^Q} = (-4\mathcal{L}_{-3} - 8\mathcal{L}_{-1}\mathcal{L}_{-2} - 2\mathcal{L}^3_{-1} - 8\mathcal{H}_{-3} - 8\mathcal{H}_{-1}\mathcal{H}_{-2} + \\ -8\mathcal{L}_{-2}\mathcal{H}_{-1} - 6\mathcal{L}^2_{-1}\mathcal{H}_{-1} - 10\mathcal{L}_{-1}\mathcal{H}_{-2} - 4\mathcal{L}_{-1}\mathcal{H}^2_{-1} + \mathcal{L}^2_{-1}\mathcal{Q}_{-1}\mathcal{G}_0 + 2\mathcal{L}_{-1}\mathcal{H}_{-1}\mathcal{Q}_{-1}\mathcal{G}_0 + \\ \mathcal{L}_{-1}\mathcal{Q}_{-1}\mathcal{G}_{-1} + 4\mathcal{H}_{-1}\mathcal{Q}_{-1}\mathcal{G}_{-1} + 3\mathcal{L}_{-1}\mathcal{Q}_{-2}\mathcal{G}_0 + 6\mathcal{Q}_{-2}\mathcal{G}_{-1} - 4\mathcal{Q}_{-1}\mathcal{G}_{-2})|-4,-2\rangle^Q,$$

$$|\chi\rangle^{(1)G}_{3,|-4,-2\rangle^Q} = |\hat{\chi}\rangle^{(0)G}_{3,|-4,-1\rangle^G} = (-4\mathcal{L}_{-3} - 8\mathcal{L}_{-1}\mathcal{L}_{-2} - 2\mathcal{L}^3_{-1} + 2\mathcal{H}_{-1}\mathcal{H}_{-2} + 2\mathcal{L}_{-1}\mathcal{H}^2_{-1} + \\ \mathcal{L}^2_{-1}\mathcal{G}_{-1}\mathcal{Q}_0 - \mathcal{H}_{-2}\mathcal{G}_{-1}\mathcal{Q}_0 - \mathcal{H}^2_{-1}\mathcal{G}_{-1}\mathcal{Q}_0 + \mathcal{L}_{-1}\mathcal{G}_{-1}\mathcal{Q}_{-1} - 3\mathcal{H}_{-1}\mathcal{G}_{-1}\mathcal{Q}_{-1} + \\ 3\mathcal{L}_{-1}\mathcal{G}_{-2}\mathcal{Q}_0 + 3\mathcal{H}_{-1}\mathcal{G}_{-2}\mathcal{Q}_0 + 6\mathcal{G}_{-2}\mathcal{Q}_{-1} - 4\mathcal{G}_{-1}\mathcal{Q}_{-2})|-4,-1\rangle^G,$$

where the two last singular vectors have been expressed as singular vectors of their equivalent types, using $|-4,-1\rangle^G = \mathcal{G}_0\,|-4,-2\rangle^Q$ and $|-4,-2\rangle^Q = \mathcal{Q}_0\,|-4,-1\rangle^G$. We see that this four-member subfamily consists of two Dörrzapf pairs: one pair of linearly independent singular vectors of type $|\chi\rangle^{(0)G}_{|\phi\rangle^G}$ (equivalently $|\chi\rangle^{(1)G}_{|\phi\rangle^Q}$) at the same level in the same Verma module , and one pair of linearly independent singular vectors of type $|\chi\rangle^{(0)Q}_{|\phi\rangle^Q}$ (equivalently $|\chi\rangle^{(-1)Q}_{|\phi\rangle^G}$) at the same level in the same Verma module. Observe that the Verma module $V(|-4,-1\rangle^G)$ has $\mathbf{h} = -1 = \frac{t}{2}$, therefore the singular vectors of the same type are transformed into each other by the simplest loop of sequence B, *i.e.* just by the mapping given by $\mathcal{G}_0\,\mathcal{A}$. This subfamily is attached further, acting with $\mathcal{U}_{\pm1}$, to the pair of singular vectors $|\chi\rangle^{(0)Q}_{3,|-3,-4\rangle^Q}$ and $|\chi\rangle^{(0)G}_{3,|-3,1\rangle^G}$, which in spite of having zero conformal weight ($\Delta + l = 0$) are not chiral, although they are in the same Verma modules at the same level than the chiral singular vectors $|\chi\rangle^{(0)G,Q}_{3,|-3,1\rangle^G}$ and $|\chi\rangle^{(0)G,Q}_{3,|-3,-4\rangle^Q}$:

$$|\chi\rangle^{(0)Q}_{3,\,|-3,-4\rangle^Q} = (-4\mathcal{L}_{-3} - 8\mathcal{L}_{-1}\mathcal{L}_{-2} - 2\mathcal{L}^3_{-1} - 8\mathcal{L}_{-2}\mathcal{H}_{-1} - 6\mathcal{L}^2_{-1}\mathcal{H}_{-1} - 2\mathcal{L}_{-1}\mathcal{H}_{-2} + \\ -4\mathcal{L}_{-1}\mathcal{H}^2_{-1} + \mathcal{L}^2_{-1}\mathcal{G}_0\mathcal{Q}_{-1} - 2\mathcal{H}_{-2}\mathcal{G}_0\mathcal{Q}_{-1} + 2\mathcal{L}_{-1}\mathcal{H}_{-1}\mathcal{G}_0\mathcal{Q}_{-1} + \mathcal{L}_{-1}\mathcal{G}_0\mathcal{Q}_{-2} + \\ -2\mathcal{H}_{-1}\mathcal{G}_0\mathcal{Q}_{-2} + 3\mathcal{L}_{-1}\mathcal{G}_{-1}\mathcal{Q}_{-1} + 6\mathcal{H}_{-1}\mathcal{G}_{-1}\mathcal{Q}_{-1} + 6\mathcal{G}_{-1}\mathcal{Q}_{-2} - 4\mathcal{G}_0\mathcal{Q}_{-3})|-3,-4\rangle^Q,$$

$$|\chi\rangle^{(0)G}_{3,\,|-3,\,1\rangle^G} = (-4\mathcal{L}_{-3} - 8\mathcal{L}_{-1}\mathcal{L}_{-2} - 2\mathcal{L}^3_{-1} - 8\mathcal{H}_{-3} + 2\mathcal{H}_{-1}\mathcal{H}_{-2} - 8\mathcal{L}_{-1}\mathcal{H}_{-2} + \\ 2\mathcal{L}_{-1}\mathcal{H}^2_{-1} + \mathcal{L}^2_{-1}\mathcal{Q}_0\mathcal{G}_{-1} + \mathcal{H}_{-2}\mathcal{Q}_0\mathcal{G}_{-1} - \mathcal{H}^2_{-1}\mathcal{Q}_0\mathcal{G}_{-1} + 3\mathcal{L}_{-1}\mathcal{Q}_{-1}\mathcal{G}_{-1} + \\ -3\mathcal{H}_{-1}\mathcal{Q}_{-1}\mathcal{G}_{-1} + \mathcal{L}_{-1}\mathcal{Q}_0\mathcal{G}_{-2} + 3\mathcal{H}_{-1}\mathcal{Q}_0\mathcal{G}_{-2} + 6\mathcal{Q}_{-1}\mathcal{G}_{-2} - 4\mathcal{Q}_0\mathcal{G}_{-3})|-3,1\rangle^G.$$

Hence we have encountered two more Dörrzapf pairs with the particularity that in each pair one of the singular vectors turns out to be chiral (chiral vectors are just particular cases of $\mathcal{G}_0$-closed vectors, as well as particular cases of $\mathcal{Q}_0$-closed vectors). We have checked that these singular vectors satisfy the Dörrzapf conditions (6.1). Observe that in this case the singular vectors of the same type are not connected to each other by a loop of sequence B, but rather by the mapping $|\chi\rangle^{(0)G}_{3,\,|-3,\,1\rangle^G} = \mathcal{A}\mathcal{U}_1\,\mathcal{G}_0\,\mathcal{U}_1\,|\chi\rangle^{(0)G,Q}_{3,\,|-3,\,1\rangle^G}$.

Acting now with $\mathcal{G}_0$ and $\mathcal{Q}_0$ one obtains *secondary* chiral singular vectors $|\chi\rangle^{(1)G,Q}_{3,\,|-3,-4\rangle^Q}$ and $|\chi\rangle^{(-1)G,Q}_{3,\,|-3,\,1\rangle^G}$ at level zero with respect to the singular vectors on which they are built. Therefore these arrows $\mathcal{G}_0$, $\mathcal{Q}_0$ cannot be reversed unlike the previous ones. One can attach arrows $\mathcal{U}_{\pm1}$ to these secondary singular vectors, and with them the four-member subfamily: $|\chi\rangle^{(-1)Q}_{4,\,|-4,-2\rangle^Q}$, $|\chi\rangle^{(1)G}_{4,\,|-4,-1\rangle^G}$, $|\chi\rangle^{(0)G,Q}_{4,\,|-4,-2\rangle^Q}$, $|\chi\rangle^{(0)G,Q}_{4,\,|-4,-1\rangle^G}$, at level 4, where the chiral singular vectors are, in turn, secondary singular vectors of the non-chiral ones. Using $\mathcal{U}_{\pm1}$ again a new subfamily is attached: $|\chi\rangle^{(0)Q}_{4,\,|-3,-4\rangle^Q}$, $|\chi\rangle^{(0)G}_{4,\,|-3,\,1\rangle^G}$, $|\chi\rangle^{(0)Q}_{4,\,|-3,\,0\rangle^Q}$, $|\chi\rangle^{(0)G}_{4,\,|-3,-3\rangle^G}$. Using $\mathcal{U}_{\pm1}$ once more one attaches the subfamily: $|\chi\rangle^{(0)Q}_{4,\,|0,-6\rangle^Q}$, $|\chi\rangle^{(0)G}_{4,\,|0,\,3\rangle^G}$, $|\chi\rangle^{(1)G}_{4,\,|0,-6\rangle^Q}$, $|\chi\rangle^{(-1)Q}_{4,\,|0,\,3\rangle^G}$. There are no further attachements to this subfamily since $\Delta = 0$, so that it is not possible to write singular vectors equivalent to $|\chi\rangle^{(1)G}_{4,\,|0,-6\rangle^Q}$ and $|\chi\rangle^{(-1)Q}_{4,\,|0,\,3\rangle^G}$ where to attach more arrows $\mathcal{U}_{\pm1}$.

The attachements of subfamilies is not finished, however, because the uncharged chiral singular vectors are equivalent to charged chiral singular vectors, and the spectral flow mappings $\mathcal{U}_{\pm1}$ distinguish betwen a given chiral singular vector and the equivalent one.

$$
\begin{array}{ccc}
\mathcal{U}_1 \uparrow & & \\
|\chi\rangle^{(-1)G,Q}_{3,\,|-3,-3\rangle^G} & & \\
\mathcal{A} \updownarrow & & \\
|\chi\rangle^{(1)G,Q}_{3,\,|-3,\,0\rangle^Q} & & \\
\mathcal{U}_1 \uparrow & & \\
|\chi\rangle^{(1)G}_{4,\,|0,\,3\rangle^G} & \xrightarrow{\mathcal{Q}_0} & |\chi\rangle^{(0)Q}_{4,\,|0,\,3\rangle^G} \\
\mathcal{A} \updownarrow & & \updownarrow \mathcal{A} \\
|\chi\rangle^{(-1)Q}_{4,\,|0,-6\rangle^Q} & \xrightarrow{\mathcal{G}_0} & |\chi\rangle^{(0)G}_{4,\,|0,-6\rangle^Q}
\end{array}
\tag{C.2}
$$

Let us move to diagram (C.2). The uncharged chiral couple of singular vectors $|\chi\rangle^{(0)G,Q}_{3,\,|-3,\,1\rangle^G}$ and $|\chi\rangle^{(0)G,Q}_{3,\,|-3,-4\rangle^Q}$, in diagram (C.1), is equivalent to the charged chiral couple $|\chi\rangle^{(1)G,Q}_{3,\,|-3,\,0\rangle^Q}$ and $|\chi\rangle^{(-1)G,Q}_{3,\,|-3,-3\rangle^G}$ in diagram (C.2). These singular vectors are attached by $\mathcal{U}_{\pm1}$ to the four-member subfamily: $|\chi\rangle^{(1)G}_{4,\,|0,\,3\rangle^G}$, $|\chi\rangle^{(-1)Q}_{4,\,|0,-6\rangle^Q}$, $|\chi\rangle^{(0)Q}_{4,\,|0,\,3\rangle^G}$, $|\chi\rangle^{(0)G}_{4,\,|0,-6\rangle^Q}$, at level 4. Since $\Delta = 0$ for this subfamily there are no further attachements.

Now let us consider the charged chiral couple $|\chi\rangle^{(1)G,Q}_{3,\,|-3,-4\rangle^Q}$ and $|\chi\rangle^{(-1)G,Q}_{3,\,|-3,\,1\rangle^G}$. It is equivalent to the uncharged chiral couple $|\chi\rangle^{(0)G,Q}_{3,\,|-3,-3\rangle^G}$ and $|\chi\rangle^{(0)G,Q}_{3,\,|-3,\,0\rangle^Q}$, that can be attached using $\mathcal{U}_{\pm1}$ to the subfamily: $|\chi\rangle^{(0)Q}_{3,\,|0,-6\rangle^Q}$, $|\chi\rangle^{(0)G}_{3,\,|0,\,3\rangle^G}$, $|\chi\rangle^{(1)G}_{3,\,|0,-6\rangle^Q}$, $|\chi\rangle^{(-1)Q}_{3,\,|0,\,3\rangle^G}$. Since $\Delta = 0$ for this subfamily there are no further attachements.

Finally let us consider the uncharged chiral couple $|\chi\rangle^{(0)G,Q}_{4,\,|-4,-1\rangle^G}$ and $|\chi\rangle^{(0)G,Q}_{4,\,|-4,-2\rangle^Q}$. It is equivalent to the charged chiral couple $|\chi\rangle^{(1)G,Q}_{4,\,|-4,-2\rangle^Q}$ and $|\chi\rangle^{(-1)G,Q}_{4,\,|-4,-1\rangle^G}$, that can be attached using $\mathcal{U}_{\pm1}$ to the subfamily: $|\chi\rangle^{(-1)Q}_{5,\,|-3,-4\rangle^Q}$, $|\chi\rangle^{(1)G}_{5,\,|-3,\,1\rangle^G}$, $|\chi\rangle^{(1)Q}_{5,\,|-3,\,0\rangle^Q}$, $|\chi\rangle^{(-1)G}_{5,\,|-3,-3\rangle^G}$, at level 5. This subfamily is in turn attached by $\mathcal{U}_{\pm1}$ to the subfamily: $|\chi\rangle^{(-1)Q}_{6,\,|0,-6\rangle^Q}$, $|\chi\rangle^{(1)G}_{6,\,|0,\,3\rangle^G}$, $|\chi\rangle^{(0)G}_{6,\,|0,-6\rangle^Q}$, $|\chi\rangle^{(0)Q}_{6,\,|0,\,3\rangle^G}$, at level 6. There are no further attachements since $\Delta = 0$ for this subfamily.

The complete family consists therefore of thirty-eight singular vectors at levels 3, 4, 5, and 6 distributed in five different Verma modules. Two of them have $\Delta = 0$ and consequently a single primary state: $V(|0,\,3\rangle^G)$ and $V(|0,-6\rangle^Q)$. The other three Verma modules have $\Delta \neq 0$ and consequently two primary states. Namely, $V(|-4,-1\rangle^G) = V(|-4,-2\rangle^Q)$, $V(|-3,\,1\rangle^G) = V(|-3,\,0\rangle^Q)$, and $V(|-3,-3\rangle^G) = V(|-3,-4\rangle^Q)$.

References

[1] S. Mukherji, S. Mukhi and A. Sen, Phys. Lett. B266 (1991) 337

[2] B.H. Lian and G.J. Zuckerman, Phys. Lett. B254 (1991) 417

[3] S. Mukhi and C. Vafa, Nucl. Phys. B407 (1993) 667

[4] L. Benoit and Y. Saint-Aubin, Phys. Lett. B215 (1988) 517;
A. Kent, Phys. Lett. B273 (1991) 56;
M. Bauer, P. di Francesco, C. Itzykson, and J.-B. Zuber, Nucl. Phys. B362 (1991) 515

[5] F.G. Malikov, B.L. Feigin, and D.B. Fuchs, Funk. An. Prilozh. 20 N2 (1986) 25

[6] M. Bauer and N. Sochen, Comm. Math. Phys. 152 (1993) 127; Phys. Lett. B275 (1992) 82

[7] L. Benoit and Y. Saint-Aubin Int. J. Mod. Phys. A7 (1992) 3023; Lett. Math. Phys. Vol. 23 (1991) 117;
C.S. Huang, D.H. Zhang and Q.R. Zheng, Nucl. Phys. B389 (1993) 81;
G.M.T. Watts, Nucl. Phys. B407 (1993) 213

[8] M. Dörrzapf, Int. J. Mod. Phys. A10 (1995) 2143

[9] M. Dörrzapf, Comm. Math. Phys. 180 (1996) 195

[10] P. Bowcock and G.M.T. Watts, Phys. Lett. B297 (1992) 282;
Z. Bajnok, Phys. Lett. B320 (1994) 36; Phys. Lett. B329 (1994) 225

[11] M. Kato and S. Matsuda, Phys. Lett. B172 (1986) 216; Advanced Studies in Pure Mathematics 16 (1988) 205

[12] A.Ch. Ganchev and V.B. Petkova, Phys. Lett. B293 (1992) 56; Phys. Lett. B318 (1993) 77

[13] P. Furlan, A.Ch. Ganchev, and V.B. Petkova, Nucl. Phys. B431 (1994) 622-666

[14] B. Gato-Rivera and A. M. Semikhatov, Phys. Lett. B293 (1992) 72, Theor. Mat. Fiz. 95 (1993) 239, Theor. Math. Phys. 95 (1993) 536

[15] B. Gato-Rivera and A. M. Semikhatov, Nucl. Phys. B408 (1993) 133

[16] A.M. Semikhatov, Mod. Phys. Lett. A9 (1994) 1867

[17] B. Gato-Rivera and J.I. Rosado, Phys. Lett. B346 (1995) 63

[18] B. Gato-Rivera and J.I. Rosado, Phys. Lett. B369 (1996) 7

[19] B. Gato-Rivera, "The Even and the Odd Spectral Flows on the N=2 Superconformal Algebras", hep-th/9707211 (1997), to be published in Nucl. Phys. B

[20] R. Dijkgraaf, E. Verlinde and H. Verlinde, Nucl. Phys. B352 (1991) 59

[21] M.B. Green, J.H. Schwarz and E. Witten, "Superstring Theory: I", Cambridge University Press 1987

[22] M. Dörrzapf and B. Gato-Rivera, work in preparation

[23] A.M. Semikhatov and I.Yu Tipunin, Int. J. Mod. Phys. A11 (1996) 4597

[24] B. Gato-Rivera and J.I. Rosado, "Chiral Determinant Formulae and Subsingular Vectors for the N=2 Superconformal Algebras", hep-th/9706041. The final version will be published in Nucl. Phys. B503 (1997) 447

[25] W. Boucher, D. Friedan and A. Kent, Phys. Lett. B172 (1986) 316

[26] S. Nam, Phys. Lett. B172 (1986) 323

[27] M. Kato and S. Matsuda, Phys. Lett. B184 (1987) 184

[28] B.L. Feigin and D.B. Fuchs, "Representations of the Virasoro algebra", Advanced Studies in Contemporary Mathematics, 7; Gordon and Breach Science Publishers, N.Y. 1990

[29] B. Gato-Rivera and J.I. Rosado, "Families of Singular and Subsingular Vectors of the Antiperiodic and Periodic N=2 Superconformal Algebras", to appear in hep-th